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Brief paper

A Kalman decomposition for possibly controllable uncertain linear systems ☆

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ABSTRACT

This paper considers the structure of uncertain linear systems building on concepts of robust unobservability and possible controllability. The paper presents a new geometric characterization of the possibly controllable states. When combined with previous geometric results on robust unobservability, the results of this paper lead to a general Kalman type decomposition for uncertain linear systems which can be applied to the problem of obtaining reduced order uncertain system models.

1. Introduction

Controllability and observability are fundamental properties of a linear system; e.g., see Antsaklis and Michel (2006). This paper extends these notions to the case of uncertain linear systems with the aim of gaining a greater understanding of the structure of uncertain linear systems when applied to problems of reduced order modelling and minimal realization.

One reason for considering the issue of controllability for uncertain systems might be to determine if a robust state feedback controller can be constructed for the system; e.g., see Petersen, Ugrinovskii, and Savkin (2000). In this case, one would be interested in the question of whether the system is "controllable" for all possible values of the uncertainty; e.g., see Basile and Marro (1992), Bhattacharyya (1983) and Conte, Perdon, and Marro (1991). Similarly, one reason for considering observability for uncertain systems might be to determine if a robust state estimator can be constructed for the system; e.g., see Petersen and Savkin (1999). In this case, one would be interested in the question of whether the system is "observable" for all possible values of the uncertainty. However, these questions of robust controllability and robust observability are not the questions being addressed in this paper.

The notions of controllability and observability are central to realization theory; e.g., see Antsaklis and Michel (2006). For example, it is known that if a linear system contains unobservable or uncontrollable states, those states can be removed in order to obtain a reduced dimension realization of the system's transfer function. From this point of view, a natural extension of the notion of controllability to the case of uncertain systems, would be to consider "possibly controllable" states which are controllable for some possible values of the uncertainty. This idea was developed in the paper Petersen (2009) for the case of uncertain linear systems with structured uncertainty subject to averaged integral quadratic constraints (IQCs). Similarly, a natural extension of the notion of observability to uncertain systems is to consider robustly unobservable states which are "unobservable" for all possible values of the uncertainty. This idea was developed in the papers Petersen (2007, 2008).

In the case that a plant is modelled by a single linear time invariant (LTI) state space model, then it is natural to remove any uncontrollable or unobservable modes from this model if it is not a minimal realization since these modes would lead to an unnecessarily complex plant model. This in turn may lead to an unnecessarily complex controller being designed. For a single LTI state space model, the removal of these modes can be carried out using the standard Kalman decomposition algorithm; e.g., see Antsaklis and Michel (2006) and Kalman (1963). However, if the plant is modelled as an uncertain system, which is a set of systems, then it is not clear how to remove the uncontrollable and unobservable modes from the uncertain system model and this indeed is one of the main contributions of this paper for the class of IQC uncertain system models being considered.

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This paper builds on concepts of "robust unobservability and "possible controllability" developed in Petersen (2007, 2009). The results presented in the paper aim to provide insight into the structure of uncertain systems as it relates to questions of realization theory and reduced dimension modelling for uncertain systems.

We define notions of robust unobservability and possible controllability in terms of certain constrained optimization problems. The notion of robust unobservability used in this paper involves extending the standard linear systems definition of the observability Gramian to the case of uncertain systems; see also Scherpen and Gray (2000). Also, the notion of possible controllability used in this paper involves extending the standard linear systems definition of the controllability Gramian to the case of uncertain systems; see also Scherpen and Gray (2000). We then apply the S-procedure (e.g., see Petersen et al., 2000) to obtain conditions for robust unobservability and possible controllability in terms of unconstrained LQ optimal control problems dependent on Lagrange multiplier parameters as in Petersen (2007, 2009). From this, we develop a geometric characterization for the set of robustly unobservable states (as in Petersen, 2008) and the set of possibly controllable states. These characterizations imply that the set of robustly unobservable states is in fact a linear subspace. Similarly, we show that the set of possibly controllable states is a linear subspace; see also Bhattacharyya (1983), in Basile and Marro (1992), and Conte et al. (1991). These characterizations lead to a Kalman type decomposition for the uncertain systems under consideration; see also Kalman (1963) and Theorem 4.3 in Chapter 3 of Antsaklis and Michel (2006). This decomposition is described in the four possible cases for which an uncertain system model can have robustly unobservable states or states which are not possibly controllable. These are the cases in which a reduced dimension uncertain system model can be obtained which retains the same set of input–output behaviours as the original model. As compared to the previous papers Petersen (2007, 2008, 2009), the results of this paper enable a complete geometrical picture to be obtained which can be applied to problems of reduced dimension modelling of uncertain linear systems. Also, the results of this paper are much more computationally tractable than the results of the papers Petersen (2007, 2009). The main assumption required in this paper as compared to the previous papers Petersen (2007, 2009) is the assumption that the uncertainty is unstructured and described by a single averaged uncertainty constraint.

One practical motivation for the results being presented occurs when an uncertain system model for a plant under consideration contains states which are robustly unobservable or not possibly controllable. In this case, the results of this paper can be used to enable a reduced order model of the uncertain plant to be obtained. This reduced order model can then be used in conjunction with standard robust control or estimation methods (e.g., see Petersen & Savkin, 1999, Petersen et al., 2000) in order to obtain a reduced dimension controller or estimator.

2. Problem formulation

We consider the linear time invariant uncertain system:

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 \xi(t);$$
$$z(t) = C_1 x(t) + D_1 u(t); \qquad y(t) = C_2 x(t) + D_2 \xi(t) \tag{1}$$

where $x \in \mathbf{R}^n$ is the *state*, $y \in \mathbf{R}^l$ is the *measured output*, $z \in \mathbf{R}^h$ is the *uncertainty output*, $u \in \mathbf{R}^m$ is the *control input*, and $\xi \in \mathbf{R}^r$ is the *uncertainty input*.

For the system (1), we define the transfer function $G(s)$ to be the transfer function from the input $\xi(t)$ to the output $y(t)$; i.e., $G(s) = C_2(sI - A)^{-1}B_2 + D_2$. Also, we define the transfer function $H(s)$ to be the transfer function from the input $u(t)$ to the output $z(t)$; i.e., $H(s) = C_1(sI - A)^{-1}B_1 + D_1$.

System uncertainty. The uncertainty in the uncertain system (1) is required to satisfy a certain "Averaged Integral Quadratic Constraint".

Averaged Integral Quadratic Constraint. Let the time interval $[0, T], T > 0$ be given and let $d > 0$ be a given positive constant associated with the system (1); see also Petersen (2007), Petersen (2009) and Savkin and Petersen (1995). We will consider sequences of uncertainty inputs $\mathcal{S} = \{\xi^1(\cdot), \xi^2(\cdot), \ldots \xi^q(\cdot)\}$. The number of elements q in any such sequence is arbitrary. A sequence of uncertainty functions of the form $\mathcal{S} = \{\xi^1(\cdot), \xi^2(\cdot), \ldots \xi^q(\cdot)\}$ is an *admissible uncertainty sequence* for the system (1) if the following conditions hold: Given any $\xi^i(\cdot) \in \mathcal{S}$ and any corresponding solution $\{x^i(\cdot), \xi^i(\cdot)\}$ to (1) defined on $[0, T]$, then $\xi^i(\cdot) \in \mathbf{L}_2[0, T]$, and

$$\frac{1}{q}\sum_{i=1}^{q}\int_0^T \left(\|\xi^i(t)\|^2 - \|z^i(t)\|^2\right) dt \le d. \tag{2}$$

The class of all such admissible uncertainty sequences is denoted Ξ.

The averaged IQC uncertainty description was introduced in Savkin and Petersen (1995) as an approach to uncertainty modelling which gives tight results in the case of structured uncertainty. The paper (Petersen, 2009) gives a more detailed explanation concerning the use of the averaged IQC uncertainty description. This paper continues to use the averaged IQC uncertainty description even though it does not consider structured uncertainties since it builds on the results of Petersen (2007), Petersen (2009) which were derived using the averaged IQC uncertainty description.

Definition 1. The *robust unobservability function* for the uncertain system (1), (2) defined on $[0, T]$ is defined as $L_o(x_0, T) \triangleq \sup_{\mathcal{S}\in\Xi} \frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y(t)\|^2 dt$ where $x(0) = x_0$ in (1).

This definition extends the standard definition of the observability Gramian for linear systems.

Definition 2. Let $\mathcal{D} \triangleq \{d : d > 0\}$. A non-zero state $x_0 \in \mathbf{R}^n$ is said to be *robustly unobservable* for the uncertain system (1), (2) defined on the time interval $[0, T]$ if $\inf_{d\in\mathcal{D}} L_o(x_0, T) = 0$. The set of all robustly unobservable states for the uncertain system (1), (2) defined on the time interval $[0, T]$ is referred to as the *robustly unobservable set* $\mathcal{U}$; i.e., $\mathcal{U} \triangleq \{x \in \mathbf{R}^n : \inf_{d\in\mathcal{D}} L_o(x, T) = 0\}$.

Definition 3. The *possible controllability function* for the uncertain system (1), (2) defined on the time interval $[0, T]$ is defined as

$$L_c(x_0, T) \triangleq \sup_{\epsilon>0}\inf_{\mathcal{S}\in\Xi}\inf_{\mathcal{U}\in\mathbf{L}_2^q[0,T]}\frac{1}{q}\sum_{i=1}^{q}\left[\frac{\|x^i(T)\|^2}{\epsilon} + \int_0^T \|u^i(t)\|^2 dt\right] \tag{3}$$

where $x(0) = x_0$ in (1).

This definition extends the standard definition of the controllability Gramian for linear systems. In particular, in the special case of systems without uncertainty, this quantity will be infinite for uncontrollable states x_0.

Definition 4. A non-zero state $x_0 \in \mathbf{R}^n$ is said to be *possibly controllable on* $[0, T]$ for the uncertain system (1), (2) if $\sup_{d\in\mathcal{D}} L_c(x_0, T) < \infty$.

This definition reduces to the definition of controllable states for systems without uncertainty; e.g., see Antsaklis and Michel (2006).