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Brief paper

Control of a bicycle using virtual holonomic constraints☆

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ABSTRACT

The paper studies the problem of making Getz's bicycle model traverse a strictly convex Jordan curve with bounded roll angle and bounded speed. The approach to solving this problem is based on the virtual holonomic constraint (VHC) method. Specifically, a VHC is enforced making the roll angle of the bicycle become a function of the bicycle's position along the curve. It is shown that the VHC can be automatically generated as a periodic solution of a scalar periodic differential equation, which we call virtual constraint generator. Finally, it is shown that if the curve is sufficiently long as compared to the height of the bicycle's center of mass and its wheel base, then the enforcement of a suitable VHC makes the bicycle traverse the curve with a steady-state speed profile which is periodic and independent of initial conditions. An outcome of this work is a proof that the constrained dynamics of a Lagrangian control system subject to a VHC are generally not Lagrangian.

1. Introduction

This paper investigates the problem of maneuvering a bicycle along a closed Jordan curve $\mathcal{C}$ in the horizontal plane in such a way that the bicycle does not fall over and its velocity is bounded. The simplified bicycle model we use in this paper, developed by Neil Getz (Getz, 1994; Getz & Marsden, 1995), views the bicycle as a point mass with a side slip velocity constraint, and models its roll dynamics as those of an inverted pendulum; see Fig. 1. The model neglects, among other things, the steering kinematics and the wheels dynamics with the associated gyroscopic effect.

In Hauser, Saccon, and Frezza (2004), the authors investigate the maneuvering problem for Getz's bicycle using a dynamic inversion approach to determine bounded roll trajectories. They constrain the bicycle on the curve and, given a desired velocity signal $v(t)$, they find a trajectory with the property that the velocity of the bicycle is $v(t)$ and its roll angle φ is in the interval $(-\pi/2, \pi/2)$, i.e, the bicycle does not fall over. In Hauser and Saccon (2006), the authors develop an algorithm to compute the minimum-time speed profile for a point-mass motorcycle compatible with the constraint that the lateral and longitudinal accelerations do not make the tires slip, and apply their algorithm to Getz's bicycle model.

The problem of maneuvering Getz's bicycle along a closed curve is equivalent to moving the pivot point of an inverted pendulum around the curve without making the pendulum fall over. On the other hand, the seemingly different problem of maneuvering Hauser's PVTOL aircraft (Hauser, Sastry, & Meyer, 1992) along a closed curve in the vertical plane can be viewed as the problem of moving the pivot of an inverted pendulum around the curve without making the pendulum fall over. The two problems are, therefore, closely related, the main difference being the fact that in the former case the pendulum lies on a plane which is orthogonal to the plane of the curve, while in the latter case it lies on the same plane. In Consolini, Maggiore, Nielsen, and Tosques (2010), the path following problem for the PVTOL was solved by enforcing a virtual holonomic constraint (VHC) which specifies the roll angle of the PVTOL as a function of its position on the curve. In this paper, we follow a similar approach for the bicycle model, and impose a VHC relating the bicycle's roll angle to its position along the curve. However, rather than finding one VHC, as we did in Consolini et al. (2010), we show how to automatically generate VHCs as periodic solutions of a scalar periodic differential equation which we call the *virtual constraint generator*. We show that if the path is sufficiently long compared to the height of the bicycle's center of mass and the wheel base, then the VHC can be chosen so that on the constraint manifold the bicycle traverses the entire curve with bounded speed, and its speed profile is periodic in steady-state. Finally, we design a controller that enforces the VHC, and recovers the asymptotic properties of the bicycle on the constraint manifold.

The concept of VHC is a promising paradigm for motion control. It is one of the central ideas in the work of Grizzle and collaborators

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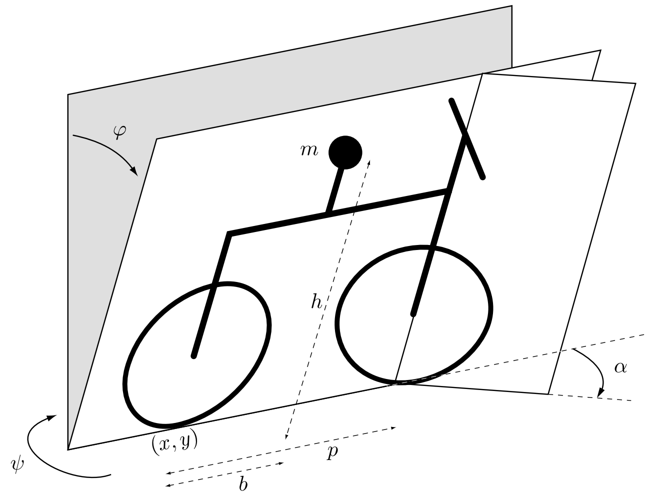

Fig. 1. Getz's bicycle model.

on biped locomotion (e.g., Plestan, Grizzle, Westervelt, & Abba, 2003; Westervelt, Grizzle, & Koditschek, 2003), where VHCs are used to encode different walking gaits. The work of Shiriaev and collaborators in Canudas-de-Wit (2004), Freidovich, Robertsson, Shiriaev, and Johansson (2008) and Shiriaev, Perram, and Canudas-de-Wit (2005) investigates VHCs for Lagrangian control systems, i.e., systems of the form Bloch (2003)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Gu, \tag{1}$$

with control input u and smooth Lagrangian $L(q, \dot{q}) = (1/2)\dot{q}^\top D(q)\dot{q} - V(q)$, with $D = D^\top$ positive definite. In Shiriaev et al. (2005), the authors consider systems of the form (1) with degree of underactuation one. They find an integral of motion for the constrained dynamics, and use it to select a desired closed orbit on the constraint manifold. This orbit is then stabilized by linearizing the control system along it, and designing a time-varying controller for the linearization. In Shiriaev, Freidovich, and Gusev (2010), these ideas are extended to systems with degree of underactuation greater than one, and in Shiriaev, Freidovich, Robertsson, Johansson, and Sandberg (2007) they are applied to the stabilization of oscillations in the Furuta pendulum. In Maggiore and Consolini (2013), we investigated VHCs for Lagrangian control systems with degree of underactuation one. We introduced and characterized a notion of regularity of VHCs, and we presented sufficient conditions under which the reduced dynamics on the constraint manifold (described by a second-order unforced system) are Lagrangian (i.e., they satisfy the Euler–Lagrange equations, which have the form (1) with zero right-hand side). An outcome of this paper (Proposition 4.1) is a simple sufficient condition under which the reduced dynamics are *not* Lagrangian. We refer the reader to Remarks 4.3, 4.4 and 4.8 for more details.

This paper is organized as follows. In Section 2 we present Getz's bicycle model and we formulate the maneuvering problem investigated in this paper. In Section 3 we present the virtual constraint generator idea. The main result is Proposition 3.3 which gives a constructive methodology to find VHCs for Getz's bicycle that meet the requirements of the maneuvering problem. In Section 4 we analyze the motion of Getz's bicycle on the virtual constraint manifold. In Proposition 4.1 we provide a general result with sufficient conditions under which an unforced second-order system of a certain form possesses an asymptotically stable closed orbit. In Proposition 4.6 we apply this result to Getz's bicycle model. In Section 5 we bring these results together and solve the maneuvering problem. Finally, in Section 6 we make remarks on numerical implementation of the proposed controller.

Notation. Throughout this paper we use the following notation. If x is a real number and $T > 0$, the number x modulo T is denoted by $[x]_T$. We let $[\mathbb{R}]_T := \{[x]_T : x \in \mathbb{R}\}$. The set $[\mathbb{R}]_T$ is diffeomorphic to the unit circle. We let $\pi : \mathbb{R} \to [\mathbb{R}]_T$ be defined as $\pi(t) = [t]_T$. Then, π is a smooth covering map (see Lee, 2013, p. 91). If M is a smooth manifold, and $h : [\mathbb{R}]_T \to M$ is a smooth function, we define $\tilde{h} := h \circ \pi : \mathbb{R} \to M$. This is a T-periodic function. Moreover, by Lee (2013, Theorem 4.29), $\tilde{h}$ is smooth if and only if h is smooth. Finally, $\mathrm{Im}(h)$ denotes the image of a function h.

2. Problem formulation

Consider the bicycle model depicted in Fig. 1, with the following variable conventions (taken from Hauser et al., 2004):

- (x, y) — coordinates of the point of contact of the rear wheel
- φ — roll angle (a positive $\dot{\varphi}$ implies that the bicycle leans to the right)
- ψ — yaw angle (a positive $\dot{\psi}$ means that the bicycle turns right)
- α — projected steering angle on the (x, y) plane
- b — distance between the projection of the center of mass and the point of contact of the rear wheel
- p — wheel base
- h — pendulum length
- v — forward linear velocity of the bicycle
- f — thrust force.

We denote $\bar{\kappa} = (\tan\alpha)/p = \dot{\psi}/v$. For a given velocity signal $v(t)$ and steering angle signal $\alpha(t)$, $\bar{\kappa}(t)$ represents the curvature of the path $(x(t), y(t))$ traced by the point of contact of the rear wheel. In Getz and Marsden (1995), the bicycle of Fig. 1 was modeled by writing the Lagrangian of the unconstrained bicycle, incorporating the nonholonomic constraints that the wheels roll without slipping in the Lagrangian, and then extracting the model through the Lagrange–d'Alembert equations as in Bloch (2003, Section 5.2). The resulting model, which we will refer to as *Getz's bicycle model*, reads as

$$\begin{aligned} &\dot{\bar{\kappa}} = \tau \\ &M \begin{bmatrix} \ddot{\varphi} \\ \dot{v} \end{bmatrix} = F + B \begin{bmatrix} \tau \\ f \end{bmatrix}, \end{aligned} \tag{2}$$

where τ, the time derivative of the curvature $\bar{\kappa}(t)$, and f are the control inputs and, denoting $s_\varphi = \sin\varphi$, $c_\varphi = \cos\varphi$,

$$M = \begin{bmatrix} h^2 & bhc_\varphi\bar{\kappa} \\ bhc_\varphi\bar{\kappa} & 1 + (b^2 + h^2 s_\varphi^2)\bar{\kappa}^2 - 2h\bar{\kappa}s_\varphi \end{bmatrix},$$

$$F = \begin{bmatrix} ghs_\varphi - (1 - h\bar{\kappa}s_\varphi)hc_\varphi\bar{\kappa}v^2 \\ (1 - h\bar{\kappa}s_\varphi)2hc_\varphi\bar{\kappa}v\dot{\varphi} + bh\bar{\kappa}s_\varphi\dot{\varphi}^2 \end{bmatrix},$$

$$B = \begin{bmatrix} -bhc_\varphi v & 0 \\ -(b^2\bar{\kappa} - hs_\varphi(1 - h\bar{\kappa}s_\varphi))v & 1/m \end{bmatrix}.$$

In this model, M is the inertia matrix, F represents the sum of Coriolis, centrifugal and conservative forces, and B is the input matrix. Now consider a C^3 closed Jordan curve $\mathcal{C}$ in the (x, y) plane with regular parametrization $\sigma : [\mathbb{R}]_T \to \mathbb{R}^2$, not necessarily unit speed. Let $\kappa : [\mathbb{R}]_T \to \mathbb{R}$ denote the signed curvature of $\mathcal{C}$. Throughout this paper, we assume the following.

Assumption 1. The curve $\mathcal{C}$ is strictly convex, i.e., $\kappa(s) > 0$ for all $s \in [\mathbb{R}]_T$.

In this paper, we investigate the dynamics of the bicycle when the point (x, y) is made to move along the curve $\mathcal{C}$ by an appropriate choice of the steering angle. In order to derive the constrained dynamics, suppose that $(x(0), y(0)) \in \mathcal{C}$, i.e., $(x(0), y(0)) = \sigma(s_0)$, for some $s_0 \in [\mathbb{R}]_T$. A point $\sigma(s(t))$ moving on $\mathcal{C}$ has linear velocity

$$v(t) = \|\sigma'(s(t))\|\dot{s}(t) \tag{3}$$