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Perturbation-based regularization for signal estimation in linear discrete ill-posed problems

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ABSTRACT

Estimating the values of unknown parameters in ill-posed problems from corrupted measured data presents formidable challenges in ill-posed problems. In such problems, many of the fundamental estimation methods fail to provide meaningful stabilized solutions. In this work, we propose a new regularization approach combined with a new regularization-parameter selection method for linear least-squares discrete ill-posed problems called constrained perturbation regularization approach (COPRA). The proposed COPRA is based on perturbing the singular-value structure of the linear model matrix to enhance the stability of the problem solution. Unlike many regularization methods that seek to minimize the estimated data error, the proposed approach is developed to minimize the mean-squared error of the estimator, which is the objective in many estimation scenarios. The performance of the proposed approach is demonstrated by applying it to a large set of real-world discrete ill-posed problems. Simulation results show that the proposed approach outperforms a set of benchmark regularization methods in most cases. In addition, the approach enjoys the shortest runtime and offers the highest level of robustness of all the tested benchmark regularization methods.

1. Introduction

We consider the standard problem of recovering an unknown signal $\mathbf{x}_0 \in \mathbb{R}^n$ from a vector $\mathbf{y} \in \mathbb{R}^m$ of noisy, linear observations given by

$$\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \mathbf{z}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a known linear-model matrix, and $\mathbf{z} \in \mathbb{R}^{m\times 1}$ is a vector of additive white Gaussian noise (AWGN) with unknown variance $\sigma_{\mathbf{z}}^2$ that is independent of $\mathbf{x}_0$. This problem has been extensively studied because of its practical and theoretical importance in many fields of science and engineering, e.g., communication, signal processing, computer vision, control theory, and economics [1–3].

Over the past years, several mathematical tools have been developed for estimating the unknown vector $\mathbf{x}_0$. The most prominent approach is the ordinary least-squares (OLS) estimator [4], which finds an estimate $\hat{\mathbf{x}}_{\text{OLS}}$ of $\mathbf{x}_0$ by minimizing the Euclidean norm of the estimator residual error, i.e.,

$$\hat{\mathbf{x}}_{\text{OLS}} = \arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2. \tag{2}$$

If $\mathbf{A}$ is a full column rank matrix, then (2) has the unique solution

$$\hat{\mathbf{x}}_{\text{OLS}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{y} = \mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{U}^T\mathbf{y}, \tag{3}$$

where $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$ is the singular value decomposition (SVD) of $\mathbf{A}$; $\mathbf{u}_i$ and $\mathbf{v}_i$ are the left and right orthogonal singular vectors, respectively, and the singular values σ_i are assumed to satisfy $\sigma_1 \geq \sigma_1 \geq \cdots \geq \sigma_n$.

Despite being a popular approach, the OLS estimator suffers when it is applied to discrete ill-posed problems. A problem is considered well-posed when its solution always exists, is unique, and depends continuously on the initial data. Ill-posed problems fail to satisfy at least one of these conditions [5]. The matrix $\mathbf{A}$ of an ill-posed problem is ill-conditioned and the computed OLS solution in (3) is potentially very sensitive to perturbations in the data such as $\mathbf{z}$ [6].

Discrete ill-posed problems arise in a variety of applications in signal processing and computer vision [7–10], computerized tomography [11], astronomy [12], image restoration and deblurring [13,14], and edge detection [15]. Interestingly, in all these applications, the data are gathered by convoluting a noisy signal with a detector [16,17]. A linear representation of such process is given by

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$$\int_{b_1}^{b_2} a(s,t)\mathbf{x}_0(t)\mathrm{d}t = \mathbf{y}_0(s) + \mathbf{z}(s) = \mathbf{y}(s), \tag{4}$$

where $\mathbf{y}_0(s)$ is the true signal, and the kernel function $a(s, t)$ represents the response. It is shown in [18] how a problem with a formulation similar to (4) fails to satisfy the well-posed conditions introduced above. The discretized version of (4) can be represented by (1).

To solve ill-posed problems, regularization methods, such as truncated SVD [19], hybrid methods [20], the covariance-shaping LS estimator [21], and the weighted LS estimator [22], are commonly used. These methods are based on leveraging additional known information into the solution of the problem and replacing the ill-posed problem with a well-posed one. This replacement should be done after carefully analyzing the ill-posed problem in terms of its physical plausibility and mathematical properties.

The most common and widely used approach is the regularized M-estimator that obtains an estimate $\hat{\mathbf{x}}$ of $\mathbf{x}_0$ by solving the convex problem

$$\hat{\mathbf{x}} := \arg\min_{\mathbf{x}} \mathcal{L}(\mathbf{y} - \mathbf{A}\mathbf{x}) + \gamma f(\mathbf{x}), \tag{5}$$

where the loss function $\mathcal{L} : \mathbb{R}^m \to \mathbb{R}$ measures the fit of $\mathbf{Ax}$ to the observation vector $\mathbf{y}$, the penalty function $f : \mathbb{R}^m \to \mathbb{R}$ establishes the structure of $\mathbf{x}$, and γ provides a balance between the two functions. Different choices of $\mathcal{L}$ and f distinguish the different estimation techniques. The most popular technique is the Tikhonov regularization [23] which is given in its simplified form by

$$\hat{\mathbf{x}}_{\text{RLS}} := \arg\min_{\mathbf{x}} ||\mathbf{y} - \mathbf{A}\mathbf{x}||_2^2 + \gamma\, ||\mathbf{x}||_2^2. \tag{6}$$

The solution to (6) is given by the regularized least-square (RLS) estimator,

$$\hat{\mathbf{x}}_{\text{RLS}} = \left(\mathbf{A}^T\mathbf{A} + \gamma \mathbf{I}_n\right)^{-1}\mathbf{A}^T\mathbf{y}, \tag{7}$$

where $\mathbf{I}_n$ is an $n \times n$ identity matrix. In general, γ is unknown and must be chosen judiciously.

Several methods have been proposed to select the value of the regularization parameter γ. These include the generalized cross validation (GCV) [24], L-curve [25,26], and quasi-optimal method [27]. A survey of regularization parameter selection methods is given in [28]. The GCV method obtains the regularizer γ by minimizing the GCV function, which suffers from a very flat minimum that is challenging to locate numerically. The L-curve method, on the other hand, is a graphical tool with a very high computational complexity. Finally, the quasi-optimal method does not take noise level into account. In general, the performance of these methods varies significantly depending on the nature of the problem.

1.1. Paper contributions

The contributions of this paper can be summarized as follows:

1. *New regularization approach*: We propose a new approach for linear discrete ill-posed problems that is based on adding an artificial perturbation matrix with a bounded norm to the model matrix $\mathbf{A}$. The objective of this artificial perturbation is to improve the singular-value structure of $\mathbf{A}$. This perturbation affects the fidelity of the model $\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \mathbf{z}$; as a result, the equality relation becomes invalid. We show that using such a modification provides a solution with better numerical stability.[1]

[1] The work presented in this paper is an extended version of [29].

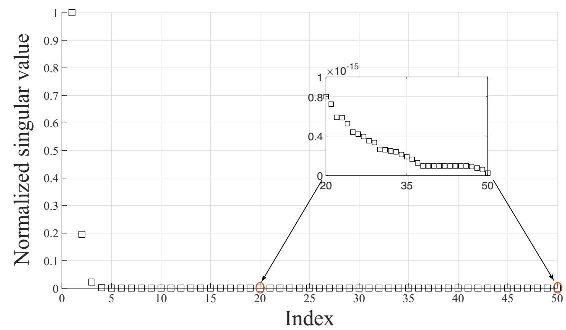

Fig. 1. Singular-value decay pattern of an ill-posed matrix, $\mathbf{A} \in \mathbb{R}^{50\times 50}$.

2. *New regularization-parameter selection method*: We develop a new approach for selecting a regularization parameter that minimizes the mean-squared error (MSE) between $\mathbf{x}_0$ and its estimate $\hat{\mathbf{x}}$, i.e., $\mathbb{E}\ ||\hat{\mathbf{x}} - \mathbf{x}_0||_2^2$.[2]
3. *Generality*: A key feature of the approach is that it does not impose any prior assumptions on $\mathbf{x}_0$. The vector $\mathbf{x}_0$ can be deterministic or stochastic and, in the later case, we do not assume any prior statistical knowledge. In addition, knowledge of the noise variance $\sigma_{\mathbf{z}}^2$ is not required. This makes the proposed approach applicable to a large number of linear discrete ill-posed problems.

1.2. Paper organization

This paper is organized as follows. In Section 2, we present the formulation of the problem and derive the solution. In Section 3, we derive the artificial perturbation bound that minimizes the MSE. Further, we derive the characteristic equation of the proposed approach which is used to obtain the regularization parameter. In Section 4, we study the properties of the characteristic equation, and in Section 5 we present the performance of the proposed approach based on simulation results. Finally, concluding remarks are given in Section 6.

1.3. Notations

Matrices are denoted by boldface uppercase letters (e.g., $\mathbf{X}$). Column vectors are represented by boldface lowercase letters (e.g., $\mathbf{x}$). The notation $(\cdot)^T$ denotes the transpose operator, $\mathbb{E}(\cdot)$ denotes the expectation operator, while $\mathbf{I}_n$ and $\boldsymbol{0}$ denote the $(n \times n)$ identity matrix and the zero matrix, respectively. The notation $||\cdot||_2$ indicates the spectral norm for matrices and the Euclidean norm for vectors. The operator diag$(\cdot)$ returns a vector that contains either the diagonal elements of a matrix, or a diagonal matrix if it operates on a vector where the diagonal entries of the matrix are the elements of that vector.

2. Proposed regularization approach

2.1. Background

We consider the linear discrete ill-posed problem in (1) without imposing any assumptions on $\mathbf{x}_0$. As stated above, matrix $\mathbf{A}$ is ill-conditioned and may have a very fast singular-value decay [31]. In Fig. 1, we observe that the singular values of matrix $\mathbf{A}$ de-

[2] Little work on MSE-based estimators is available in the literature; for example, in [30] the authors derived an estimator for the linear model problem that was based on minimizing the *worst-case* MSE (as opposed to the actual MSE) while imposing a constraint on the unknown vector $\mathbf{x}_0$.