Automatica 50 (2014) 2007–2014

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Proper discretization of homogeneous differentiators ☆

Miki Livne [1], Arie Levant

School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, 69978 Tel-Aviv, Israel

ARTICLE INFO

Article history:
Received 10 February 2013
Received in revised form
18 October 2013
Accepted 13 April 2014
Available online 23 June 2014

Keywords:
Sliding mode control
Observation
Finite-time stability

ABSTRACT

Homogeneous sliding-mode-based differentiators (HD) are known to provide for the high-accuracy robust estimation of derivatives in the presence of sampling noises and discrete measurements, provided that the differentiator dynamics evolve in continuous time. The popular one-step Euler discrete-time implementation is proved to cause differentiation accuracy deterioration, if the differentiation order exceeds 1. A novel discrete-time realization of the HD is proposed, which preserves the ultimate accuracy of the continuous-time HD also with discrete measurements.

1. Introduction

Sliding-modes (SMs) are used to control uncertain systems by keeping some functions (sliding variables) at zero due to high-frequency control switching. SMs are established in finite time, are accurate and robust (Edwards & Spurgeon, 1998; Utkin, 1992). Possible dangerous vibrations (chattering effect) constitute their main drawback (Bartolini, 1989; Edwards & Spurgeon, 1998; Fridman, 2003; Utkin, 1992).

Standard SMs (Edwards & Spurgeon, 1998; Utkin, 1992) require the sliding-variable relative degree to be 1. High-order sliding modes (HOSMs) (Bartolini, Pisano, Punta, & Usai, 2003; Levant, 1993, 1998, 2003, 2005a, 2010; Plestan, Glumineau, & Laghrouche, 2008; Shtessel & Shkolnikov, 2003) remove this restriction, placing the switching in the higher sliding-variable derivatives. Artificially increasing the relative degree one can remove the high-energy chattering (Bartolini, Pisano et al., 2003; Levant, 2010; Shtessel & Shkolnikov, 2003). Their high accuracy is due to the local homogeneity features (Levant, 2005a).

One of the main applications of sliding-mode control is the robust finite-time-exact differentiation and observation (Bartolini, Pisano, & Usai, 2000; Bejarano & Fridman, 2010; Kobayashi, Suzuki, & Furuta, 2007; Levant, 1998, 2003; Shtessel & Shkolnikov, 2003; Utkin, 1992; Yu & Xu, 1996). HOSM-based homogeneous differentiator (HD) (Levant, 2003) estimates n derivatives of a signal, provided the absolute value of its $(n+1)$th derivative has a known bound. Contrary to the popular linear (Atassi & Khalil, 2000) and nonlinear (Wang, Chen, & Yang, 2007) high-gain observers, having been robust with respect to noises, HDs also produce exact finite-time derivative estimations in the absence of noises. Such differentiators have found a lot of theoretical and practical applications (Barbot, Saadaoui, Djemai, & Manamanni, 2007; Bartolini, Damiano et al., 2003; Bartolini, Pisano et al., 2003; Defoort, Floquet, Kokosy, & Perruquetti, 2009; Imine, Fridman, Shraim, & Djemai, 2011; Iqbal, Bhatti, Ayubi, & Khan, 2011; Rabhi, M'Sirdi, Naamane, & Jaballah, 2010; Shtessel & Shkolnikov, 2003; Su, Muller, & Zheng, 2007).

The HD accuracy originates from the homogeneity of the error dynamics (Levant, 2003). It is asymptotically optimal in the presence of infinitesimal input noises (Levant, 1998), and the accuracy of its ith derivative is of the order τ^{n-i+1}, with the sampling interval τ, if the noise is absent.

The recent HD modifications (for example Angulo, Moreno, & Fridman, 2013, Cruz-Zavala, Moreno, & Fridman, 2012) contain additional higher order terms and feature faster convergence. The asymptotic accuracy is the same (Angulo, Moreno, & Fridman, 2012), if the local homogeneous error dynamics is preserved. It is usually worsened, if the local homogeneity is lost (Efimov & Fridman, 2011).

The above features were proved under the assumption that the system evolves in continuous time between the sampling time instants. Unfortunately, in practice the differentiator is a hybrid computer-based discrete dynamic system with a sampled continuous-time input. It obviously requires special study and design.

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Dragan Nešić under the direction of Editor Andrew R. Teel.

E-mail addresses: miki.livne@gmail.com (M. Livne), levant@post.tau.ac.il (A. Levant).

[1] Tel.: +972 52 4236344; fax: +972 3 6407543.

http://dx.doi.org/10.1016/j.automatica.2014.05.028
0005-1098/

A natural approach is to make the discrete-time HD (Levant, 2003) emulate the corresponding continuous-time HD. Since the system is discontinuous, the Euler method is taken with the integration step much less than the sampling period, which makes the integration step choice difficult. Hence, only one Euler integration step is usually applied at each sampling interval. The corresponding asymptotic accuracy is calculated in this paper for the cases of constant and variable sampling intervals. In particular, the accuracy is proved to be proportional to the sampling interval in the first case, whereas it is worse in the second case. Thus, the high accuracy of the continuous-time HD (Levant, 2003) is lost, if the HD order is higher than 1.

We propose a novel discretization scheme of the differentiators (Levant, 2003). Terms of higher-order with respect to the sampling intervals are added to the original Euler integration scheme. Similar terms are notably introduced in Barbot, Monaco, and Normand-Cyrot (1996) for the proper analysis of discrete dynamics. The proposed scheme preserves the computational simplicity of the one-step Euler scheme and provides for the homogeneous discrete error dynamics. Thus, the novel scheme restores the asymptotic accuracy of its continuous-time counterpart. Simulation demonstrates the calculated accuracies and the advantages of the proposed scheme.

Notation

A sum or multiplication of two sets is understood in the Minkowski sense, e.g., $AB = \{ab|\ a \in A, b \in B\}$. $d(x, A)$ is the Euclidean distance from $x \in \mathbf{R}^m$ to $A \subset \mathbf{R}^m$, $d(x, A) = \inf\{\|x - a\| \mid a \in A\}$. Following Filippov (1988), $A^\varepsilon = \{x \in \mathbf{R}^m \mid d(x, A) \le \varepsilon\}$; $\overline{\text{co}}(A)$ is the convex closure of A. Denote $f(A) = \{f(x) \mid x \in A\}$, and $F(A) = \bigcup_{x\in A} F(x)$ for any function f and set-valued function F.

The distance $d_{\text{Hs}}(A, B)$ between non-empty bounded sets A, B is taken in the Hausdorff metric, $d_{\text{Hs}}(A, B) = \max\big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\big\}$.

A set-valued function $F(x) \subset \mathbf{R}^n,\ x \in \mathbf{R}^m$, is called continuous, if $\lim_{x\to y} d_{\text{Hs}}(F(x), F(y)) = 0$, and upper-semicontinuous, if $\lim_{x\to y} (\sup\{d(z, F(y)) \mid z \in F(x)\}) = 0$.

Let $p_i = \deg x_i,\ p_i > 0$, be homogeneous degrees (weights) of the coordinates $x_1, \dots, x_m$. Then $\|x\|_h = \big(|x_1|^{p/p_1} + \cdots + |x_n|^{p/p_m}\big)^{1/p}$ is called the homogeneous norm, $p \ge \max\{p_i \mid i = 1, 2, \dots, m\}$.

2. Preliminaries: HOSM-based differentiation and discretization problem

The main idea of the differentiation based on control methods is to construct a dynamic system, tracking an input function with no knowledge of its derivatives. Let the input be $f(t) = f_0(t) + \eta(t)$, $f : \mathbf{R} \to \mathbf{R}$, where $\eta(t)$ is a Lebesgue-measurable bounded noise, $|\eta| \le \varepsilon,\ \varepsilon \ge 0$ is unknown. The function $f_0(t)$ is an n-times differentiable unknown function to be restored together with its n derivatives. The last derivative $f_0^{(n)}$ is known to have a Lipschitz constant $L > 0$, which means that $f_0^{(n+1)}(t) \in [-L, L]$ almost everywhere.

Note that the considered noise restrictions actually imply that the "worst-case" bounded noises are considered. That approach significantly differs from stochastic noise restrictions, or the requirement that the noises be "highly fluctuating" functions (Fliess, Join, & Sira-Ramírez, 2008) with infinitesimally-small integrals over any finite time interval.

A general differentiator mostly has the form

$$\begin{aligned} \dot z_i &= \varphi_i(z_0 - f) + z_{i+1}, \quad i = 0, 1, \dots, n-1, \\ \dot z_n &= \varphi_n(z_0 - f), \end{aligned} \tag{1}$$

where φ_i is a scalar function of scalar argument (Angulo et al., 2013; Atassi & Khalil, 2000; Cruz-Zavala et al., 2012; Levant, 2003). The system is understood in the Filippov sense (Filippov, 1988) to allow discontinuities of φ_i. Subtracting $f_0^{(i+1)}$ from both sides, denoting $\sigma_i = z_i - f_0^{(i)}$ and using $f_0^{(n+1)}(t) \in [-L, L]$, with $\eta = 0$ obtain

$$\begin{aligned} \dot\sigma_i &= \varphi_i(\sigma_0) + \sigma_{i+1}, \quad i = 0, 1, \dots, n-1, \\ \dot\sigma_n &\in \varphi_n(\sigma_0) + [-L, L], \end{aligned} \tag{2}$$

which is a differential inclusion in the error space $\sigma_0, \sigma_1, \dots, \sigma_n$. Here and further, for notational simplicity, the equality is considered as an inclusion with the corresponding set having only one element. Solutions of a differential inclusion are defined as absolutely continuous functions satisfying the inclusion almost everywhere.

Inclusion (2) becomes homogeneous and finite-time-stable with properly chosen functions φ_i. The homogeneity means that some positive number (called *the weight* or *the homogeneity degree* Bacciotti & Rosier, 2005) is assigned to each coordinate σ_i, $\deg\sigma_i = m_i,\ m_i > 0$. Also the time t gets its weight $\deg t = p$ (called the *minus* homogeneity degree of the inclusion Levant, 2005a), which means that the transformation

$$(t, \sigma_0, \sigma_1, \dots, \sigma_n) \mapsto (\kappa^p t, \kappa^{m_0}\sigma_0, \kappa^{m_1}\sigma_1, \dots, \kappa^{m_n}\sigma_n) \tag{3}$$

preserves the trajectories of (2) with any positive κ. Recall that a function of $\sigma_0, \sigma_1, \dots, \sigma_n$ is said to have the homogeneity degree (weight) q, if the same transformation of the arguments is equivalent to the multiplication of the function by κ^q.

Since (2) is finite-time stable, the inclusion homogeneity degree is to be negative (Levant, 2005a). It is easy to see that all weights can be proportionally changed, thus in the following assume that the homogeneity degree is -1, i.e., $\deg t = 1$. Due to the segment present in the last nth equation of (2) the only possible weight of $\dot\sigma_n$ is 0, thus $\deg\sigma_n = 1$, and $\deg\sigma_i = n - i + 1,\ i = 0, \dots, n$ (Levant, 2005a).

The recursive form of the nth-order homogeneous HOSM differentiator (Levant, 2003) is

$$\begin{aligned} \dot z_0 &= -\tilde\lambda_n L^{\frac{1}{n+1}} |z_0 - f_0|^{\frac{n}{n+1}} \operatorname{sign}(z_0 - f_0) + z_1, \\ \dot z_1 &= -\tilde\lambda_{n-1} L^{\frac{1}{n}} |z_1 - \dot z_0|^{\frac{n-1}{n}} \operatorname{sign}(z_1 - \dot z_0) + z_2, \\ &\dots \\ \dot z_n &= -\tilde\lambda_0 L \operatorname{sign}(z_n - \dot z_{n-1}). \end{aligned} \tag{4}$$

Here $z_i, i = 0, 1, \dots, n$, is the estimation of $f_0^{(i)}$, and parameters $\tilde\lambda_i$ of differentiator (4) are chosen in advance for each n. An infinite sequence of parameters $\tilde\lambda_i$ can be built, which is valid for all n (Levant, 2003). In particular, one can choose $\tilde\lambda_0 = 1.1, \tilde\lambda_1 = 1.5, \tilde\lambda_2 = 2, \tilde\lambda_3 = 3, \tilde\lambda_4 = 5, \tilde\lambda_5 = 8$ (Levant, 2005b), which correspond to the differentiators of the order $n,\ n \le 5$.

In the absence of noises the equalities $z_i = f_0^{(i)}$ are established in finite time. In the presence of a sampling noise with the maximal magnitude ε the accuracy $|z_i - f_0^{(i)}| = O(\varepsilon^{i/(n+1)})$ is obtained, and these asymptotics cannot be improved (Levant, 2003).

Extracting $\dot z_i$ from (4) obtain the standard form (1) with

$$\varphi_i(z_0 - f_0) = -\lambda_{n-i} L^{\frac{i+1}{n+1}} |z_0 - f_0|^{\frac{n-i}{n+1}} \operatorname{sign}(z_0 - f_0), \tag{5}$$

and the new coefficients $\lambda_0, \lambda_1, \dots, \lambda_k > 0$, calculated from (4). That is,

$$\begin{aligned} \dot z_0 &= -\lambda_n L^{\frac{1}{n+1}} |z_0 - f_0|^{\frac{n}{n+1}} \operatorname{sign}(z_0 - f_0) + z_1, \\ \dot z_1 &= -\lambda_{n-1} L^{\frac{2}{n+1}} |z_0 - f_0|^{\frac{n-1}{n+1}} \operatorname{sign}(z_0 - f_0) + z_2, \\ &\dots \\ \dot z_n &= -\lambda_0 L \operatorname{sign}(z_0 - f_0), \end{aligned} \tag{6}$$