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Brief paper

Optimal control of partitioned hybrid systems via discrete-time Hamilton–Jacobi theory☆

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ABSTRACT

Computational framework for optimal control of hybrid systems with a partitioned state space is presented. It is shown that necessary conditions for optimality for a discrete-time dynamic system can be solved concurrently for various boundary conditions, according to the recent development of discrete-time Hamilton–Jacobi theory. This unique property is utilized to construct computationally efficient numerical optimization of hybrid systems where discrete switching dynamics occurs at the boundary between partitions of the configuration space. A benchmark example shows that the proposed approach has substantial computational advantages compared with the existing ones.

1. Introduction

Optimal control systems based on Hamiltonian mechanics have been studied (Fernandes, 1999; Popescu, 1997). This approach exploits the geometric structures of optimal feedback control problems thoroughly, by utilizing the rich characteristics of Hamiltonian systems (Marsden & Ratiu, 1999). In particular, the recent work in Guibout and Scheeres (2006), Park, Scheeres, Guibout, and Bloch (2008) and Park, Guibout, and Scheeres (2006) provides both unique theoretical insights in solving optimal feedback control problems by using generating functions. It is shown that the two-point boundary value problem corresponding to necessary conditions for optimality can be solved by simple algebraic manipulations of the generating functions.

Computational geometric mechanics have been studied to develop numerical integrators that preserve the underlying physical properties of a dynamic system. For example, discrete-time Euler–Lagrange equations, referred to as variational integrators, are developed according to discrete Hamilton's variational principle (Marsden & West, 2001; Moser & Veselov, 1991). They are also extended to obtain discrete-time Hamiltonian systems (Lall & West, 2006; Leok & Zhang, 2011), and discrete-time Hamilton–Jacobi theory (Ohsawa, Bloch, & Leok, 2011).

Numerical flow of discrete-time mechanical systems naturally preserves their fundamental properties such as invariants, symmetries, and symplecticities (Hairer, Lubich, & Wanner, 2000). These structure-preserving properties play an important role in qualitatively accurate computation of long-term dynamics. In particular, the desirable numerical properties of discrete-time mechanical systems have been successfully adopted into optimization problems (Junge, Marsden, & Ober-Blöbaum, 2005; Lee, Leok, & McClamroch, 2009). As the numerical solutions of discrete-time mechanics are more robust and faithful, the iteration process of optimization are free of artificial numerical dissipations caused by conventional integrators.

Optimal control of a general class of hybrid systems has attracted a considerable attention. When finding optimal trajectories for hybrid systems, the task of finding an optimal discrete switching sequence often leads to combinatorial complexities. For a class of hybrid systems defined on a partitioned state space, where switchings between different dynamical regimes occur as the continuous state of the system reaches a switching surface between partitions, the location of optimal switching points naturally determines the corresponding optimal switching sequence, thereby eliminating the need to solve a combinatorial optimization problem separately (Passenberg, Sobotka, Stursberg, Buss, & Caines, 2010). In this hierarchical approach, optimal switching points in the boundary between partitions and switching times are searched

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over the solutions of optimal control problems within each partition. This requires solving boundary value problems repeatedly, and the corresponding computational load could be excessive.

In this paper, we present a computational framework to solve optimal control of hybrid systems with a partitioned state space efficiently based on the recent development of discrete-time Hamilton–Jacobi theory. The continuous-time dynamics of each partition is represented by discrete-time mechanical systems with respect to a discrete-time sequence.

First, optimal feedback control problem for a discrete-time system defined at each partition is considered. Discrete-time optimality conditions are obtained from discrete Pontryagin's principle, and the corresponding two point boundary value problems are solved via discrete-time Hamilton–Jacobi theory uniformly for several types of boundary conditions. This is useful to compute optimal feedback controls repeatedly for varying boundary conditions. Second, this desirable computational feature is applied to solve an optimal feedback control problem of hybrid systems with partitioned state spaces by using their hierarchical structures between the discrete-time optimal trajectories at a particular partition, and the discrete switching dynamics that occur at the boundary between partitions.

The proposed approach is based on combining the distinctive approaches in computational mechanics, optimization, Hamiltonian dynamics, and hybrid systems in a creative and productive manner. The main contributions of this paper are (i) constructing a new discrete-time optimal feedback control strategy that provides optimal trajectories uniformly for varying boundary conditions, and (ii) presenting highly efficient computational scheme for optimal control of hybrid systems that is applicable to optimal feedback control. The proposed method is developed for hybrid systems with a partitioned state space, but it can be readily generalized to other hybrid systems where switching dynamics autonomously. To the author's best knowledge, there has been limited study on optimal feedback control framework for hybrid systems. The preliminary results of this paper were published in Lee (2012), which deals with the formulation of discrete-time Hamilton–Jacobi theory to construct optimal feedback control. This paper is focused on constructing computationally efficient tools for optimal control of hybrid systems, and extensive benchmark result is included to clearly illustrate the substantial computational advantages of the presented approaches.

2. Discrete-time Hamiltonian mechanics

Discrete-time Hamiltonian mechanics is formulated by discretizing Hamilton's phase space variational principle (Lall & West, 2006; Leok & Zhang, 2011; Ohsawa et al., 2011). In this section, discrete-time Hamiltonian mechanics and Hamilton–Jacobi theory are summarized, and they are extended to be utilized in the subsequent development of optimal controls.

2.1. Discrete-time Hamiltonian system

Consider a Hamiltonian system evolving on a configuration space Q, where its Hamiltonian is given by $H(q,p) : \mathsf{T}^*\mathsf{Q} \to \mathbb{R}$, where $\mathsf{T}^*\mathsf{Q}$ denotes the cotangent bundle (Marsden & Ratiu, 1999). Let $\{(q_k,p_k)\}_{k=0}^N$ be a discrete curve in $\mathsf{T}^*\mathsf{Q}$, where the state variable and the momentum at the kth time step are denoted by q_k and p_k, respectively. A discrete-time Hamiltonian is an approximation to the type II generating function for the canonical transformation between (q_k,p_k) and (q_{k+1},p_{k+1}), given by

$$H_d(q_k,p_{k+1}) \approx p_{k+1}\cdot q(t_{k+1}) - \int_{t_k}^{t_{k+1}} p(t)\cdot\dot{q}(t) - H(q(t),p(t))\,dt,$$

where $q(t), p(t)$ are solution to the continuous-time Hamilton's equation satisfying boundary conditions $q(t_k) = q_k$, and $p(t_{k+1}) = p_{k+1}$.

For given $\{(q_k,p_k)\}_{k=0}^N$, we define $\mathfrak{G}_d$ as

$$\mathfrak{G}_d = p_N\cdot q_N - \sum_{k=1}^{N-1}[p_{k+1}\cdot q_{k+1} - H_d(q_k,p_{k+1})].$$

The discrete phase space variational principle states that $\delta\mathfrak{G}_d = 0$ over discrete curves with fixed boundary conditions (q_0,p_N). This yields the discrete-time Hamilton's equations:

$$q_{k+1} = D_2H_d(q_k,p_{k+1}), \qquad p_k = D_1H_d(q_k,p_{k+1}), \tag{1}$$

where D_i denotes the derivative of a function with respect to its ith argument (Leok & Zhang, 2011).

2.2. Discrete-time Hamilton–Jacobi theory

From (1), discrete-time Hamiltonian provides the relation between state variables and momenta over a single time step. This can be extended to the transformation from the current state and momentum (q_k,p_k) to their terminal values (q_N,p_N) over an arbitrary number of time steps, namely $N-k$. Canonical transformation is a change of coordinate that preserves the form of Hamilton's equations, and any canonical transformation can be described by four types of generating function, depending on the choice of independent variables (Marsden & West, 2001). As the transformation from (q_k,p_k) to (q_N,p_N) is also a canonical transformation, there exist corresponding generating functions as summarized below.

Proposition 1 (*Lee, 2012*). *Consider a discrete flow $\{(q_k,p_k)\}_{k=0}^N$ satisfying the discrete Hamilton's equations (1). Define four types of generating functions in terms of the two independent variables chosen from the boundary condition (q_k,p_k) and (q_N,p_N) as:*

$$G_1(q_k,q_N) = -\sum_{i=k}^{N-1}[p_{i+1}\cdot q_{i+1} - H_d(q_i,p_{i+1})], \tag{2}$$

$$G_2(q_k,p_N) = p_N\cdot q_N + G_1(q_k,q_N), \tag{3}$$

$$G_3(p_k,q_N) = -p_k\cdot q_k + G_1(q_k,q_N), \tag{4}$$

$$G_4(p_k,p_N) = p_N\cdot q_N - p_k\cdot q_k + G_1(q_k,q_N), \tag{5}$$

where the dependency of generating functions on t_k and t_N is not explicitly stated for simplicity. For example $G_1(q_k,q_N)$ is a shorthand for $G_1(k,N;q_k,q_N)$, which is the type I generating function for q_k at t_k and q_N at t_N. They satisfy the following equations:

$$p_k = D_1G_1(q_k,q_N), \qquad -p_N = D_2G_1(q_k,q_N), \tag{6}$$

$$p_k = D_1G_2(q_k,p_N), \qquad q_N = D_2G_2(q_k,p_N), \tag{7}$$

$$-q_k = D_1G_3(p_k,q_N), \qquad -p_N = D_2G_3(p_k,q_N), \tag{8}$$

$$-q_k = D_1G_4(p_k,p_N), \qquad q_N = D_2G_4(p_k,p_N), \tag{9}$$

which provide algebraic relations between (q_k,p_k) and (q_N,p_N).

Proposition 2 (*Lee, 2012*). *The generating functions defined at Proposition 1 satisfy the following discrete Hamilton–Jacobi equations:*

$$\begin{aligned}G_1(q_{k-1},q_N) &= G_1(q_k,q_N) - D_1G_1(q_k,q_N)\cdot q_k \\ &\quad + H_d(q_{k-1},D_1G_1(q_k,q_N)),\end{aligned} \tag{10}$$

$$\begin{aligned}G_2(q_{k-1},p_N) &= G_2(q_k,p_N) - D_1G_2(q_k,p_N)\cdot q_k \\ &\quad + H_d(q_{k-1},D_1G_2(q_k,p_N)),\end{aligned} \tag{11}$$

$$\begin{aligned}G_3(p_{k-1},q_N) &= G_3(p_k,q_N) - p_{k-1}\cdot D_1G_3(p_{k-1},q_N) \\ &\quad + H_d(-D_1G_3(p_k,q_N),p_k),\end{aligned} \tag{12}$$

$$\begin{aligned}G_4(p_{k-1},p_N) &= G_4(p_k,p_N) + p_{k-1}\cdot D_1G_4(p_{k-1},p_N) \\ &\quad + H_d(-D_1G_4(p_{k-1},p_N),p_k).\end{aligned} \tag{13}$$