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Brief paper

A unified method for optimal arbitrary pole placement☆

Robert Schmid [a,1], Lorenzo Ntogramatzidis [b], Thang Nguyen [c], Amit Pandey [d]

[a] Department of Electrical and Electronic Engineering, University of Melbourne, Parkville, VIC 3010, Australia
[b] Department of Mathematics and Statistics, Curtin University, Perth, WA 6848, Australia
[c] Department of Engineering, University of Exeter, UK
[d] Department of Mechanical and Aerospace Engineering, University of California, San Diego, USA

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ABSTRACT

We consider the classic problem of pole placement by state feedback. We offer an eigenstructure assignment algorithm to obtain a novel parametric form for the pole-placing feedback matrix that can deliver any set of desired closed-loop eigenvalues, with any desired multiplicities. This parametric formula is then exploited to introduce an unconstrained nonlinear optimisation algorithm to obtain a feedback matrix that delivers the desired pole placement with optimal robustness and minimum gain. Lastly we compare the performance of our method against several others from the recent literature.

1. Introduction

We consider the classic problem of repeated pole placement for linear time-invariant (LTI) systems in state space form

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{1}$$

where, for all $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control input. We assume that B has full column-rank and that the pair (A, B) is reachable. We let $\mathcal{L} = \{\lambda_1, \dots, \lambda_\nu\}$ be a self-conjugate set of $\nu \leq n$ complex numbers, with associated algebraic multiplicities $\mathcal{M} = \{m_1, \dots, m_\nu\}$ satisfying $m_1 + \cdots + m_\nu = n$, and $m_i = m_j$ whenever $\lambda_i = \overline{\lambda}_j$. The problem of *exact pole placement (EPP) by state feedback* is that of finding a real feedback matrix F such that

$$(A + B\,F)\,X = X\,\Lambda, \tag{2}$$

where Λ is an $n \times n$ Jordan matrix obtained from the eigenvalues of $\mathcal{L}$, including multiplicities given by $\mathcal{M}$, and X is a matrix of closed-loop eigenvectors of unit length. The matrix Λ can be expressed in the Jordan (complex) block diagonal canonical form

$$\Lambda = \text{blkdiag}(J(\lambda_1), \dots, J(\lambda_\nu)), \tag{3}$$

where each $J(\lambda_i)$ is a Jordan matrix for λ_i of order m_i and may be composed of up to g_i mini-blocks

$$J(\lambda_i) = \text{blkdiag}(J_1(\lambda_i), \dots, J_{g_i}(\lambda_i)), \tag{4}$$

where $1 \leq g_i \leq m$. We use $\mathcal{P} \stackrel{\text{def}}{=} \{p_{i,k} \mid 1 \leq i \leq \nu, 1 \leq k \leq g_i\}$ to denote the order of each Jordan mini-block $J_k(\lambda_i)$; then $p_{i,k} = p_{j,k}$ whenever $\lambda_i = \overline{\lambda}_j$. When (A, B) is reachable, arbitrary multiplicities of the closed-loop eigenvalues can be assigned by state feedback, but the possible mini-block orders of the Jordan structure of $A + BF$ are constrained by the *controllability indices* (Rosenbrock, 1970). If $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{P}$ satisfy the conditions of the Rosenbrock theorem, we say that the triple $(\mathcal{L}, \mathcal{M}, \mathcal{P})$ defines an *assignable Jordan structure* for (A, B).

In order to consider optimal selections for the feedback matrix, it is important to have a parametric formula for the set of feedback matrices that deliver the desired pole placement. In Kautsky, Nichols, and van Dooren (1985) and Schmid, Pandey, and Nguyen (2014) parametric forms are given for the case where Λ is a diagonal matrix and the eigenstructure is non-defective; this requires $m_i \leq m$ for all $m_i \in \mathcal{M}$. Parameterisations that do not impose a constraint on the multiplicity of the eigenvalues to be assigned include Bhattacharyya and de Souza (1982); Fahmy and O'Reilly (1983); however these methods require the closed-loop eigenvalues to all be distinct from the open-loop ones.

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E-mail addresses: rschmid@unimelb.edu.au (R. Schmid), L.Ntogramatzidis@curtin.edu.au (L. Ntogramatzidis), T.Nguyen-Tien@exeter.ac.uk (T. Nguyen), appandey@ucsd.edu (A. Pandey).

[1] Tel.: +61 3 83 44 66 98; fax: +61 3 8344 7412.

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The general case where $\mathcal{L}$ contains any desired closed-loop eigenvalues and multiplicities is considered in Ait Rami, Faiz, Benzaouia, and Tadeo (2009) and Chu (2007), where parametric formulae are provided for F that use the eigenvector matrix X as a parameter. Maximum generality in these parametric formulae has however been achieved at the expense of efficiency, as the square matrix X has n^2 free parameters. By contrast, methods (Bhattacharyya & de Souza, 1982; Fahmy & O'Reilly, 1983; Kautsky et al., 1985; Schmid et al., 2014) all employ parameter matrices with mn free parameters.

The first aim of this paper is to offer a parameterisation for the pole-placing feedback matrix that combines the generality of Ait Rami et al. (2009) and Chu (2007) with the efficiency of an mn-dimensional parameter matrix. We offer a parametric formula for all feedback matrices F solving (2) for any assignable $(\mathcal{L}, \mathcal{M}, \mathcal{P})$. For a given parameter matrix K, we obtain the eigenvector matrix X_K and feedback matrix F_K by building the Jordan chains from eigenvectors selected from the kernels of the matrix pencils $[A - \lambda_i I_n \quad B]$ and thus avoid the need for matrix inversions, or the solution of Sylvester matrix equations. The parameterisation will be shown to be exhaustive of all feedback matrices that assign the desired eigenstructure.

The second aim of the paper is to seek the solution to some optimal control problems. We first consider the *robust exact pole placement problem* (REPP), which involves obtaining F that renders the eigenvalues of $A+BF$ as insensitive to perturbations in A, B and F as possible. Numerous results (Chatelin, 1993) have appeared linking the sensitivity of the eigenvalues to various measures of the *condition number* of X. Another commonly used robustness measure is the *departure from normality* of the closed loop matrix $A + BF$. For the case of diagonal Λ, there has been considerable literature on the REPP, including Ait Rami et al. (2009), Byers and Nash (1989), Chu (2007), Kautsky et al. (1985), Li, Chu, and Lin (2011), Schmid et al. (2014), Tits and Yang (1996) and Varga (2000). Papers considering the REPP for the general case where $(\mathcal{L}, \mathcal{M}, \mathcal{P})$ defines an assignable Jordan structure include Ait Rami et al. (2009) and Lam, Tam, and Tsing (1997).

A related optimal control problem is the *minimum gain exact pole placement problem* (MGEPP), which involves solving the EPP problem and also obtaining the feedback matrix F that has the least gain (smallest matrix norm), which gives a measure of the control amplitude or energy required by the control action. Recent papers addressing the MGEPP with minimum Frobenius norm for F include Ataei and Enshaee (2011) and Kochetkov and Utkin (2014).

In this paper we utilise our parametric form for the matrices X and F that solve (2) to take a unified approach to the REPP and MGEPP problems, for any assignable Jordan structure. In our first method for the REPP, we seek the parameter matrix K that minimises the Frobenius condition number of X. In our second approach to the REPP, we seek the parameter matrix that minimises the departure from normality of matrix $A + BF$. Next we address the MGEPP by seeking the parameter K that minimises the Frobenius norm of F. Finally, we combine these approaches by introducing an objective function expressed as a weighted sum of robustness and gain measures, and use gradient iterative methods to seek a local minimum.

The performance of the our algorithm will be compared against the methods of Ait Rami et al. (2009), Ataei and Enshaee (2011) and Li et al. (2011) on a number of sample systems. We see that the methods introduced in this paper can achieve superior robustness while using less gain than all three of these alternative methods.

2. Arbitrary pole placement

Here we adapt the algorithm of Klein and Moore (1977) to obtain a simple parametric formula for the gain matrix F that solves the exact pole placement problem for an assignable Jordan structure $(\mathcal{L}, \mathcal{M}, \mathcal{P})$, in terms of an arbitrary parameter matrix K with mn free dimensions. We begin with some definitions.

Given a self-conjugate set of ν complex numbers $\{\lambda_1, \dots, \lambda_\nu\}$ containing σ complex conjugate pairs, we say that the set is σ-*conformably ordered* if the first 2σ values are complex while the remaining are real, and for all odd $i \le 2\sigma$ we have $\lambda_{i+1} = \overline{\lambda}_i$. For example, the set $\{10j, -10j, 2+2j, 2-2j, 7\}$ is 2-conformably ordered. For simplicity we shall assume in the following that $\mathcal{L}$ is σ-conformably ordered.

If M is a complex matrix partitioned into ν column matrices $M = [M_1 \dots M_\nu]$, we say that M is σ-*conformably ordered* if the first 2σ column matrices of M are complex while the remaining are real, and for all odd $i \le 2\sigma$ we have $M_{i+1} = \overline{M}_i$. For a σ-conformably ordered complex matrix M, we define a real matrix $\mathrm{Re}(M)$ composed of ν column matrices of the same dimensions as those of M thus: for each odd $i \in \{1, \dots, 2\sigma\}$, the ith and $i+1$-st column matrices of $\mathrm{Re}(M)$ are $\frac{1}{2}(M_i + M_{i+1})$ and $\frac{1}{2j}(M_i - M_{i+1})$ respectively, while for $i \in \{2\sigma+1, \dots, \nu\}$, the column matrices of $\mathrm{Re}(M)$ are the same as the corresponding column matrices of M. For any real or complex matrix X with $n+m$ rows, we define matrices $\overline{\pi}(X)$ and $\underline{\pi}(X)$ by taking the first n and last m rows of X, respectively. For each $i \in \{1, \dots, \nu\}$, we define the matrix pencil

$$S(\lambda_i) \stackrel{\text{def}}{=} \begin{bmatrix} A - \lambda_i I_n & B \end{bmatrix}. \tag{5}$$

We use N_i to denote an orthonormal basis matrix for the kernel of $S(\lambda_i)$. If $\lambda_{i+1} = \overline{\lambda}_i$, then $N_{i+1} = \overline{N}_i$. Since each $S(\lambda_i)$ is $n \times (n+m)$ and (A, B) is reachable, each kernel has dimension m. We let

$$M_i \stackrel{\text{def}}{=} \begin{bmatrix} A - \lambda_i I_n & B \end{bmatrix}^{\dagger}, \tag{6}$$

where † indicates the Moore–Penrose pseudo-inverse. For any matrix X we use $X(l)$ to denote the l-th column of X.

We say that a matrix K is a *compatible parameter matrix* for $(\mathcal{L}, \mathcal{M}, \mathcal{P})$, if $K \stackrel{\text{def}}{=} \mathrm{blkdiag}\{K_1, \dots, K_\nu\}$, where each K_i has dimension $m \times m_i$, and for each $i \ge 2\sigma$, K_i is a real matrix, and for all odd $i \le 2\sigma$, we have $K_{i+1} = \overline{K}_i$. Then each K_i matrix may be partitioned as

$$K_i = \begin{bmatrix} K_{i,1} & K_{i,2} & \cdots & K_{i,g_i} \end{bmatrix}, \tag{7}$$

where each $K_{i,k}$ has dimension $m \times p_{i,k}$. For $i \in \{1, \dots, \nu\}$ and $k \in \{1, \dots, g_i\}$ we build vector chains of length $p_{i,k}$ as

$$h_{i,k}(1) = N_i K_{i,k}(1), \tag{8}$$

$$h_{i,k}(2) = M_i \overline{\pi}\{h_{i,k}(1)\} + N_i K_{i,k}(2), \tag{9}$$

$$\vdots$$

$$h_{i,k}(p_{i,k}) = M_i \overline{\pi}\{h_{i,k}(p_{i,k}-1)\} + N_i K_{i,k}(p_{i,k}). \tag{10}$$

From these column vectors we construct the matrices

$$H_{i,k} \stackrel{\text{def}}{=} [h_{i,k}(1) \dots h_{i,k}(p_{i,k})] \tag{11}$$

of dimension $(n+m) \times p_{i,k}$, and

$$H_i \stackrel{\text{def}}{=} [H_{i,1} \dots H_{i,g_i}], \qquad H_K \stackrel{\text{def}}{=} [H_1 \dots H_\nu], \qquad X_K \stackrel{\text{def}}{=} \overline{\pi}\{H_K\} \tag{12}$$

of dimension $(n+m) \times m_i$, $(n+m) \times n$ and $n \times n$, respectively. Note that H_K is σ-conformably ordered, and hence we may define real matrices

$$V_K \stackrel{\text{def}}{=} \overline{\pi}\{\mathrm{Re}(H_K)\}, \qquad W_K \stackrel{\text{def}}{=} \underline{\pi}\{\mathrm{Re}(H_K)\} \tag{13}$$

of dimensions $n \times n$ and $m \times n$, respectively. We are now ready to present the main result of this paper.