Automatica 52 (2015) 35–46

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Refined instrumental variable estimation: Maximum likelihood optimization of a unified Box–Jenkins model☆

Peter C. Young

Systems and Control Group, Lancaster Environment Centre, Lancaster University, UK
Integrated Catchment Assessment and Management Centre, Australian National University College of Medicine, Biology & Environment, Canberra, ACT, Australia

ARTICLE INFO

Article history:
Received 17 March 2014
Received in revised form
28 July 2014
Accepted 22 October 2014
Available online 20 November 2014

Keywords:
System identification
Box–Jenkins model
Maximum likelihood
Optimal instrumental variable
En-bloc estimation
Recursive estimation

ABSTRACT

For many years, various methods for the identification and estimation of parameters in linear, discrete-time transfer functions have been available and implemented in widely available Toolboxes for Matlab™. This paper considers a unified *Refined Instrumental Variable* (RIV) approach to the estimation of discrete *and* continuous-time transfer functions characterized by a unified operator that can be interpreted in terms of backward shift, derivative or delta operators. The estimation is based on the formulation of a pseudo-linear regression relationship involving optimal prefilters that is derived from an appropriately unified Box–Jenkins transfer function model. The paper shows that, contrary to apparently widely held beliefs, the iterative RIV algorithm provides a reliable solution to the maximum likelihood optimization equations for this class of Box–Jenkins transfer function models and so its *en bloc* or recursive parameter estimates are optimal in maximum likelihood, prediction error minimization and instrumental variable terms.

1. Introduction

Instrumental Variable (IV) methods of parameter estimation have a long history in the statistical and control engineering literature. IV estimation has its roots in statistics and econometrics (Reiersol, 1941) and is discussed in some detail by Kendall and Stuart (1961). Some early publications in the control engineering literature include Mayne (1967), Wong, Polak, and Chen (1967) and Young (1970). Comprehensive treatments of ordinary and optimal IV methods applied to the estimation of parameters in discrete-time transfer function models then appeared almost simultaneously in two early books (Söderström & Stoica, 1983; Young, 1984). More recent papers in this general field include Garatti, Campi, and Bittanti (2006), Gilson and Van den Hof (2005), Laurain, Toth, Gilson, and Garnier (2010, 2011), Söderström (2012), Toth, Laurain, Gilson, and Garnier (2012), Wang, Zheng, and Chen (2009) and Young (2011). The present paper concerns the optimal *Refined Instrumental Variable* (RIV) approach to the unified estimation of parameters in both discrete and continuous-time transfer function models. The basic RIV algorithm was first suggested by the present author (Young, 1976) for discrete-time models and then thoroughly evaluated and extended to multivariable and continuous-time models (Jakeman & Young, 1979; Young, 1984; Young & Jakeman, 1979, 1980). Over the subsequent years, it has been developed in various ways, with recent publications on this topic including Garnier, Gilson, Young, and Huselstein (2007), Garnier and Young (2014), Gilson, Garnier, Young, and Van den Hof (2011), Young (2008, 2011) and Young, Garnier, and Gilson (2008). The present paper follows the above references and considers estimation in the time domain. Alternative IV approaches formulated in the frequency domain (see e.g. Pintelon & Schoukens, 2001) have received much less attention, although recent research (Gilson, Welsh, & Garnier, 2013) is moving in this direction.

Unlike standard IV algorithms, the RIV approach is not based on an IV modification of a linear least-squares solution to the estimation problem, or an approximate approach to prediction error minimization. Rather, as this paper will show, it is an iterative *Pseudo-Linear Regression* (PLR) algorithm that is derived directly from the conditions required for optimization of the *Maximum Likelihood* (ML) function associated with a unified *Box–Jenkins* (BJ) transfer function model. Upon convergence of this iterative procedure, therefore, its parameter estimates are optimal in maximum likelihood, prediction error minimization and instrumental variable terms. This is a rather elegant solution because it not only provides *en-bloc* estimates that maximize the likelihood function

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Juan I. Yuz under the direction of Editor Torsten Söderström.

E-mail address: p.young@lancaster.ac.uk.

http://dx.doi.org/10.1016/j.automatica.2014.10.126
0005-1098/

but it can also produce *recursive* estimates that are identical to the repeated, stage-wise *en bloc* estimates, as in linear least squares estimation. Indeed, this was one of the primary motivations for developing the algorithm in the iterative pseudo-linear form where, at each iteration, the estimates are obtained from a linear least squares solution that can be either *en-bloc* or recursive. Normally, however, the recursive estimates are only required by the user at the final iteration, where they relate to the converged, *en-bloc* parameter estimates. Here, the recursive estimates are useful for visually appraising the nature of the convergence and associated uncertainty at the final iteration, as illustrated later in the example of Section 6.1.

The first aim of the paper is to emphasize the inherent ML derivation of the RIV parameter estimation algorithm and so heighten awareness of this derivation. The second is to show how this solution is a unified one that can be applied to discrete-time *and* continuous-time transfer function models that include models defined in terms of the backward shift, δ and derivative operators. The acronym RIV will be used to refer to the general, unified algorithm, while RIVD, RIVC and RIVδ will refer to the specific operator versions. For simplicity, the associated analysis will be presented for the case of a single input, single output, stochastic system. This is, of course, easily extended to a multiple input system where the transfer functions share a common denominator; and RIV algorithms for multiple input models with different denominators in each input channel have been developed for discrete (Jakeman, Steele, & Young, 1980) and continuous-time (Garnier et al., 2007) models: these are described fully in these references, so they will not be considered here.

A third aim is to show that the iterative optimization procedure used in the standard implementation of the RIV algorithm can be considered in an entirely equivalent 'iterative updating' form of the *Gauss–Newton* (GN) kind, demonstrating again that it is based on an *implicit* prediction error minimization procedure. This GN interpretation demonstrates how the iterative optimization strategy is seeking out a local maximum of the log-likelihood cost function via prediction error minimization. As such, it provides an alternative to standard iterative prediction error minimization that is both robust under difficult estimation conditions and, as mentioned above, yields inherent recursive estimates of the model parameters because of the pseudo-linear nature of the estimation model. The paper will argue that these recursive parameter estimates can provide a useful diagnostic tool for evaluating both the identifiability of the model and the quality of the associated parameter estimates, as well as providing an obvious link with real-time recursive RIV estimation of time-variable parameters (Ljung, 1999; Young, 2011).

The next Section 2 of the paper introduces the unified BJ model; while the maximum likelihood estimation of the parameters in this model is considered in Section 3. Section 4 shows how maximum likelihood estimation of the unified BJ model can be accomplished by transforming the system and noise sub-models into pseudo-linear regression models, whose iterative estimation within the RIV framework yields maximum likelihood estimates of the full model parameters. Section 5 outlines the main aspects of the RIV algorithm and discusses its initiation, convergence and optimality in instrumental variable terms. Finally, Section 6 presents two simulation studies that reasonably exemplify the performance of the unified RIV algorithm when applied to backward shift, derivative and δ operator transfer function models.

2. The unified Box–Jenkins model

This paper is concerned with the estimation of the parameters that characterize a *Single-Input, Single-Output* (SISO), linear, time-invariant and stable transfer function model from uniformly sampled input–output data $\{u(k),\ y(k)\}$, $k = 1, 2, \ldots, N$, where the argument k denotes the kth sample from an underlying continuous-time system. In particular, let us consider the stochastic SISO transfer function model first conceived and promoted by Box and Jenkins (1970) and Box, Jenkins, and Reinsel (1994) for discrete-time systems, which can be unified and written, at any sampling instant k, in the following decomposed form[1]:

$$\text{System TF Model}: \ x(k) = \frac{B(\mu^{-1})}{A(\mu^{-1})}u(k-\tau) \tag{1a}$$

$$\text{Noise TF Model}: \quad \xi(k) = \frac{D(\mu^{-1})}{C(\mu^{-1})}e(k); \qquad e(k) = \mathcal{N}(0, \sigma^2) \tag{1b}$$

$$\text{Output Observation}: \ y(k) = x(k) + \xi(k) \tag{1c}$$

where τ is a pure time delay and μ is a unified operator that, in the present paper, can be interpreted as the forward shift operator, denoted here by z; the derivative operator, denoted here by s; or the delta operator, δ. The 'noise-free' output $x(k)$ plays an important part in the subsequent analysis and establishes the link between maximum likelihood and instrumental variable estimation. Given the possible interpretations of the unified operator μ, it is important to note that this model is informal and represents a 'snapshot' of the system at the kth sampling instant.

The model polynomials in μ that characterize the model (1) are defined as follows,

$$\begin{aligned}
A(\mu^{-1}) &= 1 + a_1\mu^{-1} + a_2\mu^{-2} + \cdots + a_n\mu^{-n} \\
B(\mu^{-1}) &= b_0 + b_1\mu^{-1} + b_2\mu^{-2} + \cdots + b_m\mu^{-m} \\
C(\mu^{-1}) &= 1 + c_1\mu^{-1} + c_2\mu^{-2} + \cdots + c_p\mu^{-p} \\
D(\mu^{-1}) &= 1 + d_1\mu^{-1} + d_2\mu^{-2} + \cdots + d_q\mu^{-q}.
\end{aligned} \tag{2}$$

Although these definitions are required for the development of the unified results and apply directly to the polynomials of the backward shift operator model, where $\mu^{-1} = z^{-1}$, the polynomials used in the subsequent development of RIVC and RIVδ algorithms, are defined in terms of μ (see later, Section 4.3.1), i.e.,

$$\begin{aligned}
A(\mu) &= \mu^n + a_1\mu^{n-1} + a_2\mu^{n-2} + \cdots + a_n \\
B(\mu) &= b_0\mu^m + b_1\mu^{m-1} + b_2\mu^{m-2} + \cdots + b_m
\end{aligned} \tag{3}$$

which does not, of course, change the model. Also, for reference in the next section,

$$\mathbf{e}(k) = \begin{bmatrix} e(1) & e(2) \ldots, e(N) \end{bmatrix}^T; \qquad \mathbf{e}(k) = \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}) \tag{4}$$

where N is the number of samples available for estimation. The structure of the above model will be denoted by the pentad $[n \quad m \quad \tau \quad p \quad q]$ and, for simplicity of exposition, the time delay τ will be set initially to zero, without any loss of generality; and the μ^{-1} argument will be dropped from the polynomials.

Finally, note that, while this unified Box–Jenkins (BJ) model assumes the stochastic white noise source $e(k)$ is normally distributed, this is not an essential requirement for the *application* of the resultant RIV algorithms, although it is essential to the optimality of the ML approach used in the derivation of the RIV algorithm that follows below in the next two sections.

[1] Note that the nomenclature used for transfer functions here is that used for RIV estimation since 1976 (Young, 1976, 2011); in this unified context, models intended for PEM estimation in Matlab™ would use $C(\mu^{-1})/D(\mu^{-1})$ for the ARMA noise model; and/or $B(\mu^{-1})/F(\mu^{-1})$ for the system model.