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The robustness of democratic consensus ☆

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ABSTRACT

In linear models of consensus dynamics, the state of the various agents converges to a value which is a convex combination of the agents' initial states. We call it democratic if in the large scale limit (number of agents going to infinity) the vector of convex weights converges to 0 uniformly.

Democracy is a relevant property which naturally shows up when we deal with opinion dynamic models and cooperative algorithms such as consensus over a network: it says that each agent's measure/opinion is going to play a negligible role in the asymptotic behavior of the global system. It can be seen as a relaxation of average consensus, where all agents have exactly the same weight in the final value, which becomes negligible for a large number of agents.

We prove that starting from consensus models described by time-reversible stochastic matrices, under some mild technical assumptions, democracy is preserved when we perturb the linear dynamics in finitely many vertices. We want to stress that the local perturbation in general breaks the time-reversibility of the stochastic matrices. The main technical assumption needed in our result is the irreducibility of the large scale limit stochastic matrix, i.e. strong connectedness of the limit network of agents, and we show with an example that this assumption is indeed required.

1. Introduction

1.1. Consensus

Many opinion dynamics models (Golub & Jackson, 2010; Jackson, 2008) and cooperative algorithms over networks like consensus (Carli, Fagnani, Speranzon, & Zampieri, 2008; Jadbabaie, Lin, & Morse, 2003; Olfati-Saber, Fax, & Murray, 2007) are mathematically represented by a stochastic matrix $P \in \mathbb{R}^{V\times V}$ where V is a finite set. Interpreting x_i as an initial belief/opinion of agent $i \in V$ on some fact or event, or a position in a physical space, linear consensus dynamics consists in replacing each opinion x_i by a weighted average of the opinion of agent i's neighbors in the network. Such dynamics may be expressed by the equation $x(t+1) = Px(t)$ where P is row-stochastic, i.e., is nonnegative with every row summing to one. Another motivation is the design or analysis of agents such as robots moving on the real line or any Euclidean space, while exchanging messages on their respective positions in a communication network. The equation $x(t+1) = Px(t)$ now describes the situation where every agent moves to a weighted average of the position of their neighbors in the network. The robots typically seek to solve the consensus problem, i.e. to all reach a common position in the space.

It is well known that under suitable assumptions on P (i.e. irreducibility and aperiodicity) there exists $\pi \in \mathbb{R}^V$ such that

$$\lim_{t\to+\infty}(P^t x)_i \to \sum_{j\in V}\pi_j x_j(0), \quad \forall i \in V. \tag{1}$$

Moreover, $\pi_i > 0$ for all $i \in V$, $\sum_i \pi_i = 1$ and $\pi^*P = \pi^*$, where π^* denotes the transpose of π and is thus a row vector.

In terms of consensus or opinion dynamics, convergence (1) means that the opinion of all agents tends to the common value $\sum_{j\in V}\pi_j x_j$ which is a convex combination of the initial opinions. For this reason, in this paper, a stochastic matrix P for which (1) holds will be called a *consensus matrix* and the relative vector π the corresponding *consensus weight vector* of P. If π is the uniform vector (i.e. $\pi_i = |V|^{-1}$ for all i), the common asymptotic value is simply the arithmetic mean of the initial beliefs; in other terms,

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all agents equally contribute to the final common belief. This uniformity condition amounts to assuming that the matrix P is doubly stochastic (also all columns sum to 1), a sufficient condition for this being that P is symmetric.

In this paper we want to consider the situation where we have a sequence $P^{(n)}$ of consensus matrices over a state space V_n of increasing cardinality corresponding to larger and larger sets of interacting agents. The corresponding consensus weight vectors will be denoted by $\pi^{(n)}$.

1.2. Democracy

The sequence $P^{(n)}$ of consensus matrices is called *democratic* if their corresponding invariant probabilities $\pi^{(n)}$ are such that $\|\pi^{(n)}\|_\infty := \max_{i\in V_n} \pi_i^{(n)} \to 0$ for $n \to +\infty$. This says that even if the initial opinion of the various agents may have a different weight on the final consensus value, still the weight of each of them becomes negligible as the number n of agents grows to ∞. This property has already been proposed in Golub and Jackson (2010) and Jackson (2008) as 'wise society' with the following interpretation. If we assume that the initial opinion of the various agents are of type $x_i = \mu + N_i$, where $\mu \in \mathbb{R}$ is the value of a parameter we want to estimate and N_i are independent noises having mean 0 and variance σ_i^2, then, the consensus point reached by applying the consensus matrix $P^{(n)}$ is given by

$$\sum_j \pi_j^{(n)} x_j = \mu + N, \quad \text{with } N = \sum_j \pi_j^{(n)} N_j.$$

If σ_i^2 are bounded from above, it follows from a straightforward variation of the weak law of large numbers (Golub & Jackson, 2010) that democracy implies that $N \to 0$ in probability when $n \to +\infty$. In wise societies agents' asymptotic belief converges to the real value of the parameter when the number of agents goes to ∞.

A very special case is when we start from a sequence $G^{(n)}$ of connected undirected graphs (with no self loops) on the set of vertices V_n and the consensus matrices $P^{(n)}$ are obtained by assigning homogeneous weights to all neighbors of an agent. Put $d_i^{(n)}$ equal to the degree in $G^{(n)}$ of the vertex i (number of edges connected to i) and define

$$P_{ij}^{(n)} = \frac{1-\tau}{d_i^{(n)}} \quad \text{for } j \text{ neighbor of } i, \qquad P_{ii}^{(n)} = \tau \tag{2}$$

while $P_{ij}^{(n)} = 0$ if $j \neq i$ is not a neighbor of i in $G^{(n)}$, where $0 \leq \tau < 1$ is a self-confidence parameter (see e.g. Frasca, Ravazzi, Tempo, & Ishii, 2013 and Friedkin & Johnsen, 1999 for other models of self-confidence, or stubbornness, in opinion dynamics). In this case we have that $\pi_i^{(n)} = d_i^{(n)}/\sum_j d_j^{(n)}$. In this context, democracy thus happens to be a rather easily checkable property only depending on the degrees of the various nodes. In particular, if graphs are regular ($d_i^{(n)}$ constant in i) the consensus weight vectors all coincide with the uniform one. More generally, if we have a uniform bound $d_i^{(n)} \leq d$ for all n and $i \in V_n$, then, clearly, $\|\pi^{(n)}\|_\infty$ goes to 0. This example is encompassed by the more general time-reversible consensus matrices which will be revised in next section. For them, an explicit characterization of the consensus weight vectors remains available so that $\|\pi^{(n)}\|_\infty$ can be estimated and democracy can easily be checked. Quite a different story is when time-reversibility is lost (e.g. sequences $P^{(n)}$ constructed as in (2) over directed graphs $G^{(n)}$): in this case there is no general technique available to characterize the consensus weights vectors and check democracy.

In Golub and Jackson (2010) the authors propose a sufficient condition for democracy (see their Theorem 1) which can be applied also to stochastic matrices which are not time-reversible. However, one of their assumptions (Property 2) never holds when the underlying sequence of graphs have a bounded degree and this rules out many interesting examples.

1.3. Robust democracy and main result

In this paper we focus on the robustness of democracy with respect to local perturbations. More precisely, we start from a democratic sequence $P^{(n)}$ defined on a sequence of nested sets V_n of nodes (i.e., $V_n \subset V_{n+1}$) and we analyze what happens to the consensus weights vectors when $P^{(n)}$ is locally perturbed. The perturbed sequence of consensus matrices $\tilde{P}^{(n)}$ coincides with $P^{(n)}$ but in a fixed finite number of rows corresponding to a subset of vertices W.

Our Theorem 2 shows that under very mild assumptions (irreducibility of the limit chains, i.e. strong connectedness of the limit graph) $\tilde{P}^{(n)}$ maintains a weak form of democracy (pointwise convergence to 0 of the consensus weight vectors). Afterwards, we focus on time-reversible chains $P^{(n)}$ and in Theorem 3 we prove that, under some technical assumptions (essentially that degrees are bounded in the associated graphs) the perturbed sequence $\tilde{P}^{(n)}$ (possibly no longer time-reversible) remains democratic. We again want to stress the fact that the sufficient conditions for democracy proposed in Golub and Jackson (2010) cannot be applied in this context as their Property 2 will never be satisfied. The proofs of these results will be probabilistic in nature interpreting $P^{(n)}$ and $\tilde{P}^{(n)}$ as transition matrices of Markov chains and the corresponding consensus weights as invariant probability vectors. Although our motivation and applications for our results lie in the field of opinion dynamics and consensus, we find the dual language of Markov chains more convenient and powerful to express the technical results and proofs.

1.4. Applications and context

From the point of view of opinion dynamics, these results essentially say that in democratic chains, no single agent or a finite group of them can unilaterally break democracy by modifying their outgoing links or weights as long as the number of links remains bounded and the graph connected.

As a more specific example, we can consider a sequence of connected undirected graphs over a nested set of vertices V_n and $P^{(n)}$ to be the corresponding consensus matrices as defined in (2). Fix now a subset $W \subseteq V_1$ and perturb $P^{(n)}$ on W by assuming that agents in W form a small community which is inclined to give more credit to each other's opinion than to people outside of W. This can be modeled by simply assuming that, for each $i \in W$, all weights $\tilde{P}_{ij}^{(n)}$ for $j \in W$ are a factor $\lambda \geq 1$ greater than weights $\tilde{P}_{ij}^{(n)}$ for $j \notin W$. The parameter λ, called 'homophily', measures the 'closure' of the community W to external influence. Our results assert that, disregarding how large λ is, democracy is preserved: in the final consensus the opinion of these agents still plays a negligible role when $|V_n| \to +\infty$. This example is treated in a more formal way in Section 2 (see Example 5).

Related perturbation problems in the context of opinion dynamics have been considered in Acemoglu, Ozdaglar, and Parandeh Gheibib (2010) where the authors study a novel gossip consensus model where a limited number of pairwise interactions are asymmetric (one of the two agents engaged in the interaction, called forceful, does not change opinion). The mean behavior of agents is governed by a stochastic matrix $\tilde{P}$ which can be represented as the perturbation of a symmetric one P (corresponding to the situation where all interactions are symmetric). Clearly, the consensus weight vector of P is the uniform one $\pi_i = N^{-1}$ where N is the number of nodes. Their main results (Theorems 5 and 6 therein)