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Brief paper

Minimizing trigonometric matrix polynomials over semi-algebraic sets ☆

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ABSTRACT

This paper addresses the problem of minimizing the minimum eigenvalue of a trigonometric matrix polynomial. The contribution is to show that, by exploiting Putinar's Positivstellensatz and introducing suitable transformations, it is possible to derive a nonconservative approach based on semidefinite programming (SDP) whose computational burden can be significantly smaller than that of an existing method recently published. Other advantages of the proposed approach include the possibility of taking into account the presence of constraints in the form of semi-algebraic sets and establishing tightness of a found lower bound.

1. Introduction

Frequency-domain methods have been playing a key role in studying control systems since many years. These methods exploit the frequency response of the system, which is the evaluation of the transfer function onto the imaginary axis (continuous-time systems) or onto the complex unit circle (discrete-time systems), see for instance Narendra and Taylor (1973) and Qiu and Zhou (2010).

A problem arising in frequency-domain methods consists of minimizing the minimum eigenvalue of a trigonometric matrix polynomial. Indeed, this problem can be met in system modeling, for instance when looking for an approximation of given systems, and in system design, for instance when imposing bounds of the frequency response of the system.

A possible way of addressing this problem in the case of a scalar variable is through the Kalman–Yakubovich–Popov lemma, see for instance Boyd, El Ghaoui, Feron, and Balakrishnan (1994). For the case of multiple variable and scalar trigonometric polynomials, a method based on sums of squares (SOS) of trigonometric polynomials has been proposed in Megretski (2003). This method has been extended to the case of trigonometric matrix polynomials in Henrion and Vyhlidal (2012) where its application to strong stability analysis is described. See also Roh, Dumitrescu, and Vandenberghe (2007) which proposes a simplified method for trigonometric polynomials in two variables and describes its application to FIR filter design.

This paper addresses the problem of minimizing the minimum eigenvalue of a trigonometric matrix polynomial. The contribution is to show that, by exploiting Putinar's Positivstellensatz and introducing suitable transformations, it is possible to derive a nonconservative approach based on semidefinite programming (SDP) whose computational burden can be significantly smaller than that of an existing method recently published. Other advantages of the proposed approach include the possibility of taking into account the presence of constraints in the form of semi-algebraic sets and establishing tightness of a found lower bound. The proposed approach is illustrated by numerical examples which also include an application in the estimation of reduced order models.

The paper is organized as follows. Section 2 introduces the problem formulation and some preliminaries. Section 3 describes the proposed results. Section 4 presents some illustrative examples. Lastly, Section 5 concludes the paper with some final remarks.

2. Preliminaries

2.1. Problem formulation

Notation: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$: sets of natural (including zero), integer, real, and complex numbers; j: imaginary unit; $\Re(A), \Im(A)$: real

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and imaginary parts of A; $\bar{A}$: complex conjugate of A; I: identity matrix (of size specified by the context); A^T, A^H: transpose and complex conjugate transpose of A; Hermitian matrix A: a matrix satisfying $A = A^H$; $\star$: corresponding block in Hermitian matrices; $A > 0, A \geq 0$: positive definite and positive semidefinite matrix A; $\lambda_{\min}(A)$: minimum real eigenvalue of A; matrix polynomial: a matrix whose entries are polynomials; $\det(A)$: determinant of A; $\otimes$: Kronecker product; $\lfloor a \rfloor$: largest integer not greater than a; $\lceil a \rceil$: smallest integer not smaller than a; $|a|$: magnitude of a; $\mathrm{mod}(a, b)$: modulo between a and b.

Let us denote the unit circle in $\mathbb{C}$ as

$$\mathcal{T} = \{z \in \mathbb{C} : \ |z| = 1\} . \tag{1}$$

We say that $F : \mathcal{T}^n \to \mathbb{C}^{m\times m}$ is a trigonometric matrix polynomial if

$$F(z) = \sum_{k\in\mathcal{S}} F_k z^k \tag{2}$$

where $\mathcal{S}$ is a given finite subset of $\mathbb{Z}^n$, $F_k \in \mathbb{C}^{m\times m}$ for $k \in \mathcal{S}$ are given matrices, and the notation z^k stands for

$$z^k = \prod_{l=1}^{n} z_l^{k_l}. \tag{3}$$

Moreover, we say that the trigonometric matrix polynomial $F(z)$ is Hermitian over $\mathcal{T}^n$ if

$$F(z) = F(z)^H \quad \forall z \in \mathcal{T}^n. \tag{4}$$

Problem. Let $F : \mathcal{T}^n \to \mathbb{C}^{m\times m}$ and $g_i : \mathcal{T}^n \to \mathbb{C}$, $i = 1, \ldots, n_G$, be Hermitian trigonometric matrix polynomials over $\mathcal{T}^n$. The problem is to solve

$$\mu^* = \min_{z\in\mathcal{G}} \lambda_{\min}(F(z)) \tag{5}$$

where $\mathcal{G} \subseteq \mathcal{T}^n$ is the semi-algebraic set

$$\mathcal{G} = \left\{z \in \mathcal{T}^n : \ g_i(z) \geq 0 \ \forall i = 1, \ldots, n_G\right\} . \quad \square \tag{6}$$

2.2. SOS matrix polynomials

Here we briefly define SOS matrix polynomials and explain how they can be investigated via LMIs. See also Chesi (2010), Chesi, Tesi, Vicino, and Genesio (1999), Hol and Scherer (2004), Kojima (2003), Lasserre (2001), Parrilo (2000), Prajna, Papachristodoulou, and Wu (2004), Scherer and Hol (2006) and references therein for details. For reasons that will become clear in the next section, we consider matrix polynomials in $2n$ variables of size $m \times m$.

Let us start by considering the real case. Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{m\times m}$ be a matrix polynomial. We say that $A(v)$, $v \in \mathbb{R}^{2n}$, is SOS if there exist matrix polynomials $A_i : \mathbb{R}^{2n} \to \mathbb{R}^{m\times m}$, $i = 1, \ldots, k$, such that

$$A(v) = \sum_{i=1}^{k} A_i(v)^T A_i(v). \tag{7}$$

A necessary and sufficient condition for establishing whether $A(v)$ is SOS can be obtained via an LMI feasibility test. Indeed, $A(v)$ can be expressed as

$$A(v) = (I \otimes b(v))^T \left(C + L(\alpha)\right) (I \otimes b(v)) \tag{8}$$

where $b(v)$ is a vector of monomials in v, C is a symmetric matrix, and $L(\alpha)$ is a linear parametrization of the linear set

$$\mathcal{L} = \left\{\tilde{L} = \tilde{L}^T : \ (I \otimes b(v))^T \tilde{L} (I \otimes b(v)) = 0\right\} \tag{9}$$

with α free real vector. The representation (8) is known as square matrix representation (SMR) and extends the Gram matrix method for (scalar) polynomials to the matrix case. One has that $A(v)$ is SOS if and only if there exists α satisfying the LMI

$$C + L(\alpha) \geq 0. \tag{10}$$

Next, let us consider the complex case. Let $A : \mathbb{R}^{2n} \to \mathbb{C}^{m\times m}$ be a matrix polynomial. We say that $A(v)$ is SOS if there exist matrix polynomials $A_i : \mathbb{R}^{2n} \to \mathbb{C}^{m\times m}$, $i = 1, \ldots, k$, such that

$$A(v) = \sum_{i=1}^{k} A_i(v)^H A_i(v). \tag{11}$$

Similarly to the real case, this condition holds if and only if there exists α satisfying the LMI

$$\begin{pmatrix} \Re(C + L(\alpha)) & \Im(C + L(\alpha)) \\ \star & \Re(C + L(\alpha)) \end{pmatrix} \geq 0 \tag{12}$$

where C and $L(\alpha)$ are Hermitian and satisfy (8), in particular $L(\alpha)$ (with α free real vector) is a linear parametrization of the linear set in (9) where $\tilde{L}$ is Hermitian instead of symmetric.

3. Main results

3.1. Proposed approach

Let us express $F(z)$ as in (2), and define its degree as

$$\deg(F) = \max_{\substack{k\in\mathcal{S} \\ F_k\neq 0}} \sum_{l=1}^{n} |k_l|. \tag{13}$$

Assumption 1. For all $i = 1, \ldots, n_G$, $\deg(g_i)$ is even. $\square$

Let us observe that Assumption 1 can be introduced without loss of generality. Indeed, if $\deg(g_i)$ is odd for some i, one can redefine such a $g_i(z)$ as

$$g_i(z) \to g_i(z)c_i(z) \tag{14}$$

where $c_i(z) : \mathcal{T}^n \to \mathbb{C}$ is any trigonometric polynomial such that $\deg(c_i) = 1$ and

$$\forall z \in \mathcal{T}^n \ \begin{cases} c_i(z) = c_i(z)^H \\ c_i(z) > 0 \end{cases} \tag{15}$$

which ensures that the newly defined trigonometric polynomial has even degree and $\mathcal{G}$ is not modified.

Let us write $z \in \mathcal{T}^n$ as

$$z = x + jy \tag{16}$$

where $x, y \in \mathbb{R}^n$, and define $v \in \mathbb{R}^{2n}$ as

$$v = (x^T, y^T)^T. \tag{17}$$

Let us express $F(z)$ as in (2), and introduce the matrix polynomial

$$\Delta(F, v) = \sum_{k\in\mathcal{S}} F_k \prod_{\sigma=1}^{n} \delta(v, k, \sigma) \tag{18}$$

where

$$\delta(v, k, \sigma) = \begin{cases} (v_\sigma + jv_{\sigma+n})^{k_\sigma} & \text{if } k_\sigma \geq 0 \\ (v_\sigma - jv_{\sigma+n})^{k_\sigma} & \text{otherwise.} \end{cases} \tag{19}$$

Let us define the set

$$\mathcal{D} = \begin{cases} \mathbb{R} & \text{if } \Delta(F, v), \Delta(g_1, v), \ldots, \Delta(g_{n_G}, v) \text{ are real} \\ \mathbb{C} & \text{otherwise.} \end{cases} \tag{20}$$

Let $A : \mathbb{R}^{2n} \to \mathcal{D}^{m\times m}$ be a matrix polynomial expressed as

$$A(v) = \sum_{k\in\mathbb{N}^{2n}} A_k v^k \tag{21}$$

for some $A_k \in \mathcal{D}^{m\times m}$, and introduce

$$\Theta(A, v) = \sum_{k\in\mathbb{N}^{2n}} A_k \prod_{\sigma=1}^{2n} \theta(v, k, \sigma) \tag{22}$$