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Brief paper

A region-dependent gain condition for asymptotic stability☆

Humberto Stein Shiromoto[a], Vincent Andrieu[c,d], Christophe Prieur[b]

[a] INCAS3 Dr. Nassaulaan 9, 9401 HJ Assen, The Netherlands

[b] GIPSA-lab, Grenoble Campus, 11 rue des Mathématiques, BP 46, 38402 Saint Martin d'Hères Cedex, France

[c] LAGEP, Université de Lyon, Rue Victor Grignard, CPE, Bât. G, 69622 Villeurbanne, France

[d] Fachbereich C - Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, Gaußstraße 20, 42097 Wuppertal, Germany

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ABSTRACT

A sufficient condition for the stability of a system resulting from the interconnection of dynamical systems is given by the small gain theorem. Roughly speaking, to apply this theorem, it is required that the gains composition is continuous, increasing and upper bounded by the identity function. In this work, an alternative sufficient condition is presented for the case in which this criterion fails due to either lack of continuity or the bound of the composed gain is larger than the identity function. More precisely, the local (resp. non-local) asymptotic stability of the origin (resp. global attractivity of a compact set) is ensured by a region-dependent small gain condition. Under an additional condition that implies convergence of solutions for almost all initial conditions in a suitable domain, the almost global asymptotic stability of the origin is ensured. Two examples illustrate and motivate this approach.

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1. Introduction

The use of nonlinear input–output gains for stability analysis was introduced in Zames (1966) by considering a system as an input–output operator. The condition that ensures stability, called Small Gain Theorem, of interconnected systems is based on the contraction principle.

The work (Sontag, 1989) introduces a new concept of gain relating the input to system states. This notion of stability links Zames' and Lyapunov's approaches (Sontag, 2001). Characterizations in terms of dissipation and Lyapunov functions are given in Sontag and Wang (1995).

In Jiang, Teel, and Praly (1994), the contraction principle is used in the input-to-state stability notion to obtain an equivalent Small Gain Theorem. A formulation of this criterion in terms of Lyapunov functions may be found in Jiang and Mareels (1996).

Besides stability analysis, the Small Gain Theorem may also be used for the design of dynamic feedback laws satisfying robustness constraints. The interested reader is invited to see Freeman and Kokotović (2008), Sastry (1999) and references therein. Other versions of the Small Gain theorem do exist in the literature, see Angeli and Astolfi (2007), Astolfi and Praly (2012), Ito (2006) and Ito and Jiang (2009) for not necessarily ISS systems.

In order to apply the Small Gain Theorem, it is required that the composition of the nonlinear gains is smaller than the argument for all of its positive values. Such a condition, called Small Gain Condition, restricts the application of the Small Gain Theorem to a composition of well chosen gains.

In this work, an alternative criterion for the stabilization of interconnected systems is provided when a single Small Gain Condition does not hold globally. It consists in showing that if the two conditions hold: (1) a local (resp. non-local) Small Gain Condition holds in a local (resp. non-local) region of the state space, and the intersection of the local and non-local is empty, and (2) outside the union of these regions, the set of initial conditions from which the associated trajectories do not converge to the local region has measure zero, then the resulting interconnected system is almost asymptotically stable (this notion is precisely defined below). In this paper, a sufficient condition guaranteeing this property to hold is presented. Moreover, for planar systems, an extension of Bendixson's criterion to regions which are not simply connected is given. This allows to obtain global asymptotic stability of the origin.

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E-mail addresses: humberto.shiromoto@ieee.org (H. Stein Shiromoto), vincent.andrieu@gmail.com (V. Andrieu), christophe.prieur@gipsa-lab.fr (C. Prieur).

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This approach may be seen as a unification of two small gain conditions that hold in different regions: a local and a non-local. The use of a unifying approach for local and non-local properties is well known in the literature see Andrieu and Prieur (2010) in the context of control Lyapunov functions, see Chaillet, Angeli, and Ito (2012) when uniting iISS and ISS properties.

This paper is organized as follows. In Section 2, the system under consideration and the problem statement are presented. Section 3 states the assumptions to solve the problem under consideration and the main results. Section 4 presents examples that illustrate the assumptions and main results. In Section 5 the proofs of the main results are presented. Section 6 collects some concluding remarks.

Notation. Let $k \in \mathbb{Z}_{>0}$. Let $\mathbf{S}$ be a subset of $\mathbb{R}^k$ containing the origin, the notation $\mathbf{S}_{\neq 0}$ stands for $\mathbf{S} \setminus \{0\}$. The *closure* of $\mathbf{S}$ is denoted by $\mathtt{cl}\{\mathbf{S}\}$. Let $x \in \mathbb{R}^k$, the notation $|x|$ stands for *Euclidean norm* of x. An *open* (resp. *closed*) *ball centered at* $x \in \mathbb{R}^k$ *with radius* $r > 0$ is denoted by $\mathbf{B}_{<r}(x)$ (resp. $\mathbf{B}_{\leq r}(x)$). A continuous function $f : \mathbb{R}^k \to \mathbb{R}$ is *positive definite* if, for every $x \in \mathbb{R}^k \setminus \{0\}$, $f(x) > 0$ and $f(0) = 0$. It is *proper* if $|f(x)| \to \infty$, as $|x| \to \infty$. By $\mathcal{L}^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^k)$ the class of functions $\eta : \mathbb{R} \to \mathbb{R}^k$ that are *locally essentially bounded*. By $\mathcal{C}^s$ it is denoted the class of *s-times continuously differentiable functions*, by $\mathcal{P}$ it is denoted the class of *positive definite functions*, by $\mathcal{K}$ it is denoted the class of *continuous, positive definite and strictly increasing functions* $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$; it is denoted by $\mathcal{K}_\infty$ if, in addition, they are *unbounded*. Let $c \in \mathbb{R}_{>0}$, the notation $\Omega_{\diamond c}(f)$ stands for the *subset of* $\mathbb{R}^k$ defined by $\{x \in \mathbb{R}^k : f(x) \diamond c\}$, where $\diamond$ is a *comparison operator* (i.e., $=$, $<$, $\geq$ etc). The *support* of the function f is the set $\mathtt{supp} := \{x \in \mathbb{R}^k : f(x) \neq 0\}$. By $\mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$ it is denoted the class of functions $g : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$ that are locally essentially bounded. Let $x, \bar{x} \in \mathbb{R}_{\geq 0}$, the notation $x \nearrow \bar{x}$ (resp. $x \searrow \bar{x}$) stands for $x \to \bar{x}$ with $x < \bar{x}$ (resp. $x > \bar{x}$).

2. Background and problem statement

Consider the system

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where, for every $t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$, and $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, for some positive integers n and m. Also, $f \in \mathcal{C}^1(\mathbb{R}^{n+m}, \mathbb{R}^n)$. A solution of (1) with initial condition x, and input u at time t is denoted by $X(t, x, u)$. From now on, arguments t will be omitted, and assume that the origin is *input-to-stable stable* (ISS for short) *for* (1). For further details on this concept, the interested reader is invited to consult Sontag (2001) or Sontag and Wang (1996).

A locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for which there exist $\underline{\alpha}_x, \overline{\alpha}_x \in \mathcal{K}_\infty$ such that, for every $x \in \mathbb{R}^n$, $\underline{\alpha}_x(|x|) \leq V(x) \leq \overline{\alpha}_x(|x|)$ is called *storage function*.

Inspired by Dashkovskiy, Ruffer, and Wirth (2010) and Liberzon, Nešić, and Teel (2014), the following notion of derivative will be used.

Definition 1. Consider the function $\xi : [a, b) \to \mathbb{R}$, the limit at $t \in [a, b)$

$$D^+\xi(t) = \limsup_{\tau \searrow 0} \tfrac{\xi(t+\tau)-\xi(t)}{\tau}$$

(if it exists) is called *Dini derivative*. Let k_1 and k_2 be positive integers, $(y_1, y_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, functions $\varphi : \mathbb{R}^{k_1+k_2} \to \mathbb{R}$, $h_1 : \mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$ and $h_2 : \mathbb{R}^{k_2} \to \mathbb{R}^{k_2}$. The limit

$$D^+_{h_1,h_2}\varphi(y_1, y_2) = \limsup_{\tau \searrow 0} \tfrac{\varphi(y_1+\tau h_1(y_1), y_2+\tau h_2(y_2))-\varphi(y_1,y_2)}{\tau}$$

(if it exists) is called *Dini derivative of* φ *in the* h_1 *and* h_2*-directions at* (y_1, y_2).[1] $\triangleleft$

If, for a given storage function V, there exist a proper function $\lambda_x \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, and $\alpha_x \in \mathcal{K}_\infty$ called *ISS-Lyapunov gain* such that, for every $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$|x| \geq \alpha_x(|u|) \Rightarrow D^+_f V(x, u) \leq -\lambda_x(x), \tag{2}$$

then V is called *ISS-Lyapunov function* for (1). As in Dashkovskiy et al. (2010), the proof that the existence of an ISS-Lyapunov implies that (1) is ISS goes along the lines presented in Sontag and Wang (1995).

Consider the system[2]

$$\dot{z} = g(v, z), \tag{3}$$

where $v \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, $z \in \mathbb{R}^m$, and $g \in \mathcal{C}^1(\mathbb{R}^{n+m}, \mathbb{R}^m)$. From now on, assume that $W : \mathbb{R}^{n+m} \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function for (3) with $\lambda_z \in (\mathcal{C}^0 \cap \mathcal{P})(\mathbb{R}^m, \mathbb{R}_{\geq 0})$, and $\alpha_z \in \mathcal{K}_\infty$ satisfying, for every $(v, z) \in \mathbb{R}^{n+m}$,

$$W(z) \geq \alpha_z(|v|) \Rightarrow D^+_g W(v, z) \leq -\lambda_z(z). \tag{4}$$

System under consideration. Interconnecting systems (1) and (3) yields the system

$$\begin{cases} \dot{x} = f(x, z), \\ \dot{z} = g(x, z). \end{cases} \tag{5}$$

Using the vectorial notation $y = (x, z)$, system (5) is denoted by $\dot{y} = h(y)$. A solution initiated from y in $\mathbb{R}^{n+m}$ and evaluated at time t is denoted $Y(t, y)$. The two ISS-Lyapunov inequalities (2) and (4) can be rephrased as follows. For every couple $(x, z) \in \mathbb{R}^{n+m}$,

$$\begin{aligned} V(x) \geq \gamma(W(z)) &\Rightarrow D^+_f V(x, z) \leq -\lambda_x(x), \\ W(z) \geq \delta(V(x)) &\Rightarrow D^+_g W(x, z) \leq -\lambda_z(z) \end{aligned} \tag{6}$$

with suitable functions $\gamma, \delta \in \mathcal{K}_\infty$.

A sufficient condition that ensures the stability of (5) is given by the small gain theorem (Jiang & Mareels, 1996). Roughly speaking if,

$$\forall s \in \mathbb{R}_{>0}, \qquad \gamma \circ \delta(s) < s, \tag{7}$$

then the origin is globally asymptotically stable for (5).

Problem statement. At this point, it is possible to explain the problem under consideration. ISS systems for which (7) does not hold in a bounded set of $\mathbb{R}_{\geq 0}$ are considered. This paper shows that by merging small gain arguments in different regions of the state space and employing some tools from measure theory, a sufficient condition ensuring almost global asymptotic stability of the origin is possible to be given. For planar interconnected systems, by using an extension of Bendixon's criterion, global asymptotic stability of the origin may be established.

3. Assumptions and main results

Assumption 1. There exist constant values $0 \leq \underline{M} < \overline{M} \leq \infty$ and $0 \leq \underline{N} < \overline{N} \leq \infty$, and class $\mathcal{K}_\infty$ functions γ and δ such that, for every $(x, z) \in \mathbf{S} \subset \mathbb{R}^n \times \mathbb{R}^m$, the implications

$$V(x) \geq \gamma(W(z)) \Rightarrow D^+_f V(x, z) \leq -\lambda_x(x) \tag{8}$$

$$W(z) \geq \delta(V(x)) \Rightarrow D^+_g W(x, z) \leq -\lambda_z(z) \tag{9}$$

hold, where

$$\begin{aligned} \mathbf{S} := \{&(x, z) \in \mathbb{R}^n \times \mathbb{R}^m : \underline{M} \leq V(x) \leq \overline{M}, \\ &W(z) \leq \overline{N}\} \cup \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m : \\ &V(x) \leq \overline{M}, \underline{N} \leq W(z) \leq \overline{N}\}, \quad \triangleleft \end{aligned} \tag{10}$$

[1] When the Dini derivative is taken in only one direction, the subscript denotes only such a direction.

[2] A solution of (3) with initial condition z, and input v at time t is denoted by $Z(t, z, v)$.

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