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Generating random variates for stable sub-Gaussian processes with memory

A. Mahmood [a,*], M. Chitre [a,b]

[a] Acoustic Research Laboratory, Tropical Marine Science Institute, National University of Singapore, 119227 Singapore
[b] Department of Electrical and Computer Engineering, Block E4, 4 Engineering Drive 3, 117583 Singapore

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ABSTRACT

We present a computationally efficient method to generate random variables from a univariate conditional probability density function (PDF) derived from a multivariate α-sub-Gaussian (αSG) distribution. The approach may be used to sequentially generate variates for sliding-window models that constrain immediately adjacent samples to be αSG random vectors. We initially derive and establish various properties of the conditional PDF and show it to be equivalent to a Student's t-distribution in an asymptotic sense. As the αSG PDF does not exist in closed form, we use these insights to develop a method based on the rejection sampling (accept-reject) algorithm that allows generating random variates with computational ease.

1. Introduction

The heavy-tailed *stable* distribution has been extensively used in the literature to model impulsive data [11,6,15]. Heavy-tailed distributions assign non-negligible probabilities to outliers and therefore offer good fits to amplitude distributions of impulsive datasets [11]. The motivation for employing stable models stems from the generalized central limit theorem (GCLT) [21], which states that the sum of independent and identically distributed (IID) random variables (vectors) converges to a *stable* distribution as the number of elements in the sum approaches infinity [15,21]. The GCLT is in fact the central limit theorem (CLT) but with the power constraint removed. The last statement essentially implies that the well-known Gaussian distribution is part of the stable family. Moreover, it is the only member to have light (exponential) tails [15,21]. As the validity of Gaussian models is primarily attributed to the CLT [17], the GCLT offers a similar argument for heavy-tailed stable models when the process is impulsive in nature [11,15]. From an engineering perspective, there are a number of practical scenarios where the ambient noise process is known to be impulsive. The warm shallow underwater channel, powerlines and interference-prone wireless networks are a few examples where such noise is prevalent [6,23].

In the literature, impulsive noise processes are typically assumed to be white, i.e., the samples are symmetric IID random variables [6,5,2]. Though such assumptions offer mathematical tractability in terms of developing optimized routines, they are far from realistic [3,10,7,4]. In practice, noise is seldom white and therefore has memory. Thus, at any given time, the current noise sample depends on a number of previous samples. To incorporate this dependence within the noise framework, several models have been proposed in the literature [3,10,7,6]. Recently, the *stationary α-sub-Gaussian noise with memory order m* (αSGN(m)) process was proposed to model the ambient noise in warm shallow underwater environments [7]. As highlighted by its name, the αSGN (m) model is based on the α-sub-Gaussian (αSG) distribution, which is a subclass of the stable family with the added characteristic of being symmetric and elliptical as well [14]. The new model is particularly adept in tracking the dependence between samples of the noise process whilst constraining the amplitude distribution to be that of a symmetric α-stable (SαS) random variable and is shown to outperform contemporary colored and white models in this regard.

Generating random variates is an important component of simulation-based performance analysis of systems, schemes and algorithms. The αSGN(m) model is based on a sliding-window type framework and constrains samples within the window to be αSG [7]. Therefore, a noise sample is returned from a *univariate conditional distribution* of a *multivariate* αSG distribution. As shown later, this is a computationally demanding task, especially

* Corresponding author.
E-mail addresses: tmsahme@nus.edu.sg (A. Mahmood), mandar@nus.edu.sg (M. Chitre).

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when a large number of variates are required. However, by expressing the conditional density in a suitable form and taking advantage of its properties, it is possible to compute realizations in a less time-consuming manner.

The primary contribution of this paper is to offer a way to *efficiently* generate realizations for αSG processes that are based on a sliding-window framework. The αSGN(m) model falls within this category. Generating *independent* outcomes from multivariate αSG distributions is a straightforward task and may be accomplished with computational ease [9,14]. However, extending this efficiency to processes with memory is not trivial. To accomplish this, we investigate the properties of a univariate conditional distribution derived from a (general) multivariate αSG probability density function (PDF). We show that this converges to a Student's t-distribution in an asymptotic sense and express the latter's parameters in terms of the formers. These properties are then exploited to find suitable majorizing functions which are used to efficiently generate random variates by employing the established rejection sampling (accept-reject) method [19]. Further still, the general αSG PDF cannot be expressed in closed-form [14]. For every instance of its argument, the PDF is evaluated by numerically integrating over a *heavy-tailed function* [11,14]. The same holds true for its corresponding conditional PDFs. We investigate the underlying function and show that it can be evaluated by a one-time tabulation and interpolation over a certain range, after which limiting expressions may be applied. Employing this routine in conjunction with the optimized setting of the rejection sampling algorithm substantially reduces the time taken for generating the realizations. It takes approximately a second to sequentially generate 10,000 samples of αSGN(m) on a 3.70 GHz processor.

This paper is organized as follows: in Section 2, we briefly summarize stable, SαS and multivariate αSG distributions. We then derive the univariate conditional density for the latter and comment on its properties in Section 3. Using these insights, we discuss random number generation from the conditional PDF via rejection sampling in Section 4. We wrap up by presenting the conclusions in Section 5.

2. Summary of concepts and notation

2.1. Stable distributions

A *random vector* $\vec{X} \in \mathbb{R}^d$ is stable if and only if

$$a_1\vec{X}^{(1)} + a_2\vec{X}^{(2)} \overset{d}{=} b\vec{X} + c, \tag{1}$$

where $a_1, a_2, b, c \in \mathbb{R}$, $\vec{X}^{(i)}$ are IID copies of $\vec{X}$ and $\overset{d}{=}$ implies equality in distribution [15,21]. One of the implications of (1) is that the distribution of $\vec{X}$ is *conserved* under linear transformations up to location and scale. This is termed as the stability property and is unique to this family of distributions [11,21]. From (1), one notes that if $\vec{X}$ is stable, then any permutation of $\vec{X}$ is stable as well. Further still, if $\vec{X} = [\vec{X}_1^{\mathrm{T}}, \vec{X}_2^{\mathrm{T}}]^{\mathrm{T}}$ such that $\vec{X}_1 \in \mathbb{R}^m$ and $\vec{X}_2 \in \mathbb{R}^{d-m}$, then the marginal random vectors $\vec{X}_1$ and $\vec{X}_2$ are stable too [21].

A *univariate* stable distribution is parameterized by four parameters, namely the characteristic exponent $\alpha \in (0, 2]$, the skew parameter $\beta \in [-1, +1]$, the scale $\delta \in (0, +\infty)$ and the location $\mu \in (-\infty, +\infty)$ [15,21]. In notational form, it is denoted by $\mathcal{S}(\alpha, \beta, \delta, \mu)$ [11]. Stable distributions are generally heavy-tailed, with the heaviness solely determined by α. As the latter increases, the tails become *increasingly lighter*, ultimately converging to a Gaussian distribution with mean μ and variance $2\delta^2$, i.e., $\mathcal{N}(\mu, 2\delta^2)$ for $\alpha = 2$ [15]. If $\vec{X}$ is a stable random vector, then its elements are stable random variables and have the same α.

A *univariate* SαS distribution is stable, but with β and μ set to zero [11,21]. Consequently, its PDF symmetric about zero for any α. We denote such distributions by $\mathcal{S}(\alpha, \delta)$. For $\alpha = 2$, $\mathcal{S}(2, \delta) \overset{d}{=} \mathcal{N}(0, 2\delta^2)$. Extending this to the multidimensional case, we note that $\vec{X} \in \mathbb{R}^d$ is SαS if it satisfies (1) and its PDF $f_{\vec{X}}(\mathbf{x})$ is symmetric, i.e.,

$$f_{\vec{X}}(\mathbf{x}) = f_{\vec{X}}(-\mathbf{x}), \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^d$ is a sample outcome of $\vec{X}$. Moreover, all marginal distributions of $\vec{X}$ are SαS and each element of $\vec{X}$ is SαS with the same α [11,21].

One disadvantage of employing stable random variables is the general lack of closed-form PDFs. Exceptions are the Gaussian ($\alpha = 2$) case and the Cauchy case ($\alpha = 1$). Evaluating the PDF at a point requires calling a numerical routine, which is computationally taxing [11,21]. This is more prominent in multivariate cases. However, in some instances, certain conducive properties may be exploited to reduce evaluation times. The αSG distribution is such an example and is discussed next.

2.2. The αSG distribution

An αSG distribution is *heavy-tailed* SαS with the added constraint of being *elliptical* as well [14,21]. More precisely, the random vector $\vec{X} \in \mathbb{R}^d$ is αSG ($\alpha \neq 2$) if it can be expressed as

$$\vec{X} = \sqrt{A}\,\vec{G}, \tag{3}$$

where $A \sim \mathcal{S}(\frac{\alpha}{2}, 1, 2(\cos(\frac{\pi\alpha}{4}))^{2/\alpha}, 0)$ and $\vec{G} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ is a d-dimensional Gaussian random vector with the all-zero location vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}$ [14,21,11]. Moreover, A and $\vec{G}$ are independent. The matrix $\boldsymbol{\Sigma}$ is also called the *shape matrix* of $\vec{X}$ [14]. The *marginal distribution* corresponding to any tuple of elements in $\vec{X}$ is also an αSG distribution and is therefore elliptic as well [14]. Denoting the i^{th} diagonal element of $\boldsymbol{\Sigma}$ as Σ_{ii}, the distribution of the i^{th} element in $\vec{X}$ is $\mathcal{S}(\alpha, \sqrt{\Sigma_{ii}})$.

Before we present the PDF of $\vec{X}$, it is pertinent to define the *standard isotropic* αSG vector

$$\vec{Y} = \boldsymbol{\Sigma}^{-1/2}\vec{X} = \sqrt{A}\,\vec{G}_Y, \tag{4}$$

where $\vec{G}_Y = \boldsymbol{\Sigma}^{-1/2}\vec{G} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$ and $\mathbf{I}_d$ is the $d \times d$ identity matrix. The lower-triangular matrix $\boldsymbol{\Sigma}^{1/2}$ arises from the Cholesky decomposition of $\boldsymbol{\Sigma}$, i.e., $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2}(\boldsymbol{\Sigma}^{1/2})^{\mathrm{T}}$, as the latter is a symmetric positive semi-definite matrix [18,22]. In turn, the PDF of $\vec{Y}$ can be written in terms of the PDF of the *radial* random variable $R = \|\vec{Y}\| = (\vec{Y}^{\mathrm{T}}\vec{Y})^{1/2}$ [14]. The latter can be expressed as