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Short communication

Optimum time delay estimation for complex-valued stationary signals

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ABSTRACT

The problem of finding the time delay between complex-valued sequences received at two spatially separated sensors is addressed. Considering white signal and noise processes, three delay estimation algorithms based on the minimum mean square error criterion are devised. Their asymptotic mean and mean square error expressions are derived which show that all are unbiased estimators with performance approaching the Cramér–Rao lower bound. Numerical results are included to validate the theoretical development and contrast the three techniques.

1. Introduction

Time delay estimation (TDE) between signals received at an array of sensors has been an important research topic because of its diverse applications in areas including radar, sonar, communications, acoustics, seismology, optics and biomedical engineering [1–7]. Although the general setup is to find the delays when the sensor number is at least 3 [4,8,9], we focus on the fundamental problem in this paper, namely, estimating the time-difference-of-arrival (TDOA) between two received signals. Considering the passive TDE signal model [1], the discrete-time observations can be expressed as:

$$r_1[n] = s[n] + q_1[n], \quad r_2[n] = \alpha s[n-D] + q_2[n], \quad n = 0, 1, \ldots, N-1, \tag{1}$$

where $s[n]$ is the unknown signal-of-interest, α is the attenuation constant and $D \in \mathbb{R}$ is the TDOA, while $q_1[n]$ and $q_2[n]$ are additive zero-mean noise processes. The task is to find D using the N samples of $r_1[n]$ and $r_2[n]$ received at two separated sensors.

When $s[n]$ is deterministic, such as a sinusoid [3] or chirp [5], the Fourier transform based approach can provide optimum delay estimation performance in the sense that its mean square error (MSE) attains the Cramér–Rao lower bound (CRLB) at sufficiently high signal-to-noise ratio (SNR) conditions. On the other hand, for random $s[n]$, the standard solutions include locating the maximum of the cross-correlation function between $r_1[n]$ and $r_2[n]$ [2,10], and delay modeling via a finite impulse response (FIR) filter [11–13]. Recently, the FIR filtering methodology has also been extended to handle the case when $q_1[n]$ and $q_2[n]$ are impulsive processes with the use of robust techniques [7,14]. Nevertheless, TDE for complex-valued signals is less addressed in the literature. In [15], deterministic real-valued signals are converted to the complex analytic form prior to the delay estimation process, while [16] considers noncircular observations. To the best of our knowledge, existing time delay estimation algorithms for complex signals cannot achieve the highest estimation accuracy. In this work, we contribute to the development of optimum TDE algorithms for white complex-valued data.

The rest of this paper is organized as follows. Under the umbrella of the minimum mean square error (MMSE) framework, we present three TDOA estimation approaches in Section 2. Mean and MSE as well as computational complexity of the proposed methods are studied in Section 2.1. In particular, the MSE performance of the proposed approaches can attain the CRLB asymptotically. Numerical examples for evaluation and validation are given in Section 4. Finally, we conclude our work in Section 5.

2. Algorithm development

The MMSE criterion for TDE is to minimize the following cost function:

$$J_{\mathrm{MMSE}}(\tilde{\alpha}, \tilde{D}) = \mathbb{E}\{|r_2[n] - \tilde{\alpha} r_1[n-\tilde{D}]|^2\}, \tag{2}$$

where $\tilde{\alpha}$ and $\tilde{D}$ are the optimization variables for α and D,

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respectively, and $\mathbb{E}$ represents the expectation operator. Assuming that $s[n]$, $q_1[n]$ and $q_2[n]$ are independent zero-mean white complex-valued processes with variances σ_s^2, $\sigma_{q_1}^2$ and $\sigma_{q_2}^2$, respectively, the MMSE estimates of α and D, denoted by $\hat{\alpha}$ and $\hat{D}$, are:

$$\hat{\alpha} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_{q_1}^2}\alpha, \quad \hat{D} = D, \tag{3}$$

implying that unbiased delay estimation is achieved.

For simplicity but without loss of generality, we assume $\alpha \in \mathbb{R}$. Expanding $J_{\text{MMSE}}(\tilde{\alpha}, \tilde{D})$ and noting that $\mathbb{E}\{|r_1[n-\tilde{D}]|^2\} = \sigma_s^2 + \sigma_{q_1}^2$, it is seen that $\hat{D}$ can also be obtained from:

$$\begin{aligned}\hat{D} &= \arg\max_{\tilde{D}} R_{2,1}(\tilde{D}), \quad R_{2,1}(\tilde{D}) \\ &= \mathbb{E}\{r_2^*[n] r_1[n-\tilde{D}] + r_2[n] r_1^*[n-\tilde{D}]\} \\ &= 2\mathbb{E}\{\Re\{r_2^*[n] r_1[n-\tilde{D}]\}\},\end{aligned} \tag{4}$$

where $\Re$ denotes the real part and * is the conjugate operator. Eq. (4) corresponds to the cross-correlation method as $R_{2,1}(\tilde{D}) \in \mathbb{R}$ measures the similarity between $r_2[n]$ and $r_1[n-\tilde{D}]$. To produce $r_1[n-\tilde{D}]$ from $r_1[n]$, we make use of the interpolation formula [11]:

$$r_1[n-D] = \sum_{i=-\infty}^{\infty} r_1[n-i]\operatorname{sinc}(i-D) \approx \sum_{i=-P}^{P} r_1[n-i]\operatorname{sinc}(i-D), \tag{5}$$

where $\operatorname{sinc}(v) = \sin(\pi v)/(\pi v)$ is the sinc function. Note that $P > |D|$ should be chosen sufficiently large to reduce the delay modeling error [12].

2.1. $\hat{D}_1$

Nevertheless, direct implementation of (4) requires varying $\tilde{D}$ to search for the peak in $\hat{R}_{2,1}(\tilde{D})$, which is the estimate of $R_{2,1}(\tilde{D})$ based on finite number of samples. To avoid doing so, an alternative is to employ $R_{2,1}(p)$, $p = -P, -P+1, \ldots, P$, which is easily computed. Using (5), we straightforwardly obtain:

$$R_{2,1}(p) = 2\alpha\sigma_s^2 \operatorname{sinc}(p-D), \quad p = -P, -P+1, \ldots, P. \tag{6}$$

In practice, $R_{2,1}(p)$ is replaced by its estimate using finite samples [9]:

$$\hat{R}_{2,1}(p) = \frac{2}{N-|p|}\sum_{n=0}^{N-|p|-1} \Re\{r_2^*[n] r_1[n-p]\}, \quad p = -P, -P+1, \ldots, P. \tag{7}$$

According to [12], the TDOA estimate based on sinc interpolation of $\{\hat{R}_{2,1}(p)\}$, denoted by $\hat{D}_1$, is:

$$\hat{D}_1 = \arg\max_{\tilde{D}} J_1(\tilde{D}), \quad J_1(\tilde{D}) = \sum_{p=-P}^{P} \hat{R}_{2,1}(p)\operatorname{sinc}(p-\tilde{D}). \tag{8}$$

2.2. $\hat{D}_2$

Ideally, replacing $\hat{R}_{2,1}(p)$ by $R_{2,1}(p)$ of (6) and considering $P \to \infty$ in (8), we can easily obtain $\hat{D}_1 = D$. However, $\hat{D}_1$ is biased for finite P [12]. To circumvent the delay bias, we utilize (6) and propose a least squares (LS) fit which minimizes the following cost function [13]:

$$J_2(\tilde{\gamma}, \tilde{D}) = \sum_{p=-P}^{P}\left(\hat{R}_{2,1}(p) - \tilde{\gamma}\operatorname{sinc}(p-\tilde{D})\right)^2, \tag{9}$$

where $\tilde{\gamma}$ is the optimization variable for $\gamma = 2\alpha\sigma_s^2$. As γ is easily solved with a closed-form expression from (9), we can remove $\tilde{\gamma}$ in the LS cost function. As a result, the corresponding TDOA estimate, denoted by $\hat{D}_2$, is calculated as:

$$\begin{aligned}\hat{D}_2 &= \arg\min_{\tilde{D}} J_2(\tilde{D}), \quad J_2(\tilde{D}) \\ &= \sum_{p=-P}^{P}\left(\hat{R}_{2,1}(p) - \frac{\sum_{i=-P}^{P}\hat{R}_{2,1}(i)\operatorname{sinc}(i-\tilde{D})}{\sum_{i=-P}^{P}\operatorname{sinc}^2(i-\tilde{D})}\operatorname{sinc}(p-\tilde{D})\right)^2.\end{aligned} \tag{10}$$

2.3. $\hat{D}_3$

As it is not practical to generate a perfect $r_1[n-\tilde{D}]$, our second approach is to model $\tilde{\alpha} r_1[n-\tilde{D}]$ in (2) using an FIR filter with transfer function $W(z) = \sum_{p=-P}^{P} w_p z^{-p}$ [11–13]. Using (5), the MMSE solution is then derived as

$$\begin{aligned}w_p &= \arg\min_{\tilde{w}_p} \mathbb{E}\left\{\left|r_2[n] - \sum_{p=-P}^{P}\tilde{w}_p r_1[n-p]\right|^2\right\} \\ &= \frac{\sigma_s^2}{\sigma_s^2 + \sigma_{q_1}^2}\alpha\operatorname{sinc}(p-D).\end{aligned} \tag{11}$$

With only finite number of observations, sample correlation functions in the form of (7) are used in (11), which means that the MMSE criterion is replaced by the LS regression. The LS estimate of w_p, denoted by $\hat{w}_p$, is then computed from:

$$\begin{bmatrix}\hat{w}_{-P} \\ \hat{w}_{-P+1} \\ \vdots \\ \hat{w}_P\end{bmatrix} = \begin{bmatrix}\hat{R}_{1,1}(0) & \hat{R}_{1,1}(-1) & \cdots & \hat{R}_{1,1}(-2P) \\ \hat{R}_{1,1}(1) & \hat{R}_{1,1}(0) & \cdots & \hat{R}_{1,1}(-2P+1) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{R}_{1,1}(2P) & \hat{R}_{1,1}(2P-1) & \cdots & \hat{R}_{1,1}(0)\end{bmatrix}^{-1} \begin{bmatrix}\hat{R}_{2,1}(-P) \\ \hat{R}_{2,1}(-P+1) \\ \vdots \\ \hat{R}_{2,1}(P)\end{bmatrix}, \tag{12}$$

where $\hat{R}_{1,1}(p)$ denotes the modified auto-correlation function of $r_1[n]$ and its definition aligns with (4). Following (9)–(10), which corresponds to a second LS regression step, the resultant delay estimate, denoted by $\hat{D}_3$, is:

$$\begin{aligned}\hat{D}_3 &= \arg\min_{\tilde{D}} J_3(\tilde{D}), \quad J_3(\tilde{D}) \\ &= \sum_{p=-P}^{P}\left(\hat{w}_p - \frac{\sum_{i=-P}^{P}\hat{w}_i\operatorname{sinc}(i-\tilde{D})}{\sum_{i=-P}^{P}\operatorname{sinc}^2(i-\tilde{D})}\operatorname{sinc}(p-\tilde{D})\right)^2.\end{aligned} \tag{13}$$

It is worth mentioning that all the derivations are valid for signals sampled at or over the Nyquist rate because the development is based on (5).