Signal Processing 130 (2017) 323–336

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Characteristic function based parameter estimation of skewed alpha-stable distribution: An analytical approach

Mohammadreza Hassannejad Bibalan [a], Hamidreza Amindavar [a,*], Maryam Amirmazlaghani [b]

[a] Department of Electrical Engineering, Amirkabir University of Technology, Tehran, Iran
[b] Department of Computer Engineering and Information Technology, Amirkabir University of Technology, Tehran, Iran

ARTICLE INFO

Article history:
Received 3 December 2015
Received in revised form
22 June 2016
Accepted 19 July 2016
Available online 21 July 2016

Keywords:
α-stable distribution
Parameter estimation
Characteristic function
Skewness
SAR signal

ABSTRACT

This paper introduces a new technique for analytical parameter estimation of skewed α-stable distribution with $1 < \alpha \le 2$. Stable distribution as a four-parameter non-Gaussian distribution is completely characterized by its characteristic function (CF). There are some serious limitations in parameter estimation of α-stable distribution due to the lack of closed-form expression for the general α-stable probability density function (PDF). The proposed estimator uses a hierarchical framework based on the skewed α-stable CF, and hence, allows a rapid estimation of parameters with high accuracy in real-time signal processing algorithms. In our scheme, only two values of α-stable CF, which has analytic formula, are utilized to estimate the parameters of α-stable density. In addition, the closed-form expression for estimating the required values of CF is derived. To provide a precise quantitative assessment, our proposed approach is compared with three other state-of-the-art estimators which have analytic formulas through a series of goodness-of-fit tests. Simulation results also demonstrate that the proposed method has a good accuracy both for the symmetric and non-symmetric (skewed) α-stable distributions. Furthermore, the advantage of the proposed CF based method becomes more evident through the experimental results obtained from the high-resolution SAR images.

1. Introduction

Stable distributions were first introduced by Paul Lévy in the study of sums of identical independent distributed (i.i.d.) terms in the 1920s [1]. It has been suggested that among all the heavy-tailed distributions, the family of α-stable distributions provide a substantially accurate model for impulsive interference [2]. Stable distributions as a generalization of the Gaussian distributions play an important role in SAR signals, biomedical, underwater and atmospheric environment signal processing in the presence of impulsive interference [3,4]. In recent years, there has been a growing interest in estimating the parameters of α-stable distributions [5]; however, many of these methods consider only the symmetric α-stable distributions [6]. Classical parameter estimation techniques for stable distributions are classified into six main categories [7]; quantile method (QM), characteristic function method (CFM) [8], maximum likelihood (ML) method, extreme value method (EVM), fractional lower order moment (FLOM) method [9], method of log-cumulant (MOLC) [10].

QM is one of the first estimation methods proposed by Fama and Roll [11] based on the α-stable percentiles to estimate the α-stable parameters directly from the observed data. They recognized certain patterns in tabulated quantiles of α-stable distributions and proposed estimates of α, γ and δ for symmetric α-stable distributions. McCulloch [12] extended these methods to the general skewed distributions, eliminated the bias and obtained consistent estimators for all four parameters in terms of five sample quantiles (the 5th, 25th, 50th, 75th and 95th percentiles). These estimators are characterized with a series of tabulated functions from which the estimated parameter values are given. Unfortunately, McCulloch's tables do not support the whole parameters space and are limited in accuracy, especially for the case of the location parameter [13]. Also, it is important to note that QM requires look up tables, linear interpolation and calculation of the five quantiles. Search within a lookup table and linear interpolation are not serious problems for modern machines; but, computation of the quantiles is strongly dependent to the sample size and may impose computational burden in terms of execution time [14]. If the data set is stable and if the sample set is large enough, the QM method gives reliable estimates of α-stable parameters.

* Corresponding author.
E-mail addresses: hassannejad@aut.ac.ir (M. Hassannejad Bibalan), hamidami@aut.ac.ir (H. Amindavar), mazlaghani@aut.ac.ir (M. Amirmazlaghani).

http://dx.doi.org/10.1016/j.sigpro.2016.07.020
0165-1684/

Since closed-form expressions are known for the CFs of stable laws, several researchers introduced parameter estimation methods based on the empirical characteristic functions (ECF) [15]. There are also estimation methods based on the inverse Fourier transform, specifically, in [16] a fast estimation procedure is proposed. However, such a spectral method is influenced by aliasing as well, and is not practicable for values of $\alpha < 1.2$. A numerical search algorithm based on ML approach is yet another approach for the α-stable parameter estimation [17]. Accordingly, most of estimators may be used to provide an initial guess to the search [18]. Nevertheless, a fully comprehensive search over the whole α-stable parameters space is slow and time consuming when compared to other classical distributions. For this reason, ML approach is not often an appropriate candidate for estimating the parameters of α-stable densities [19]. From all known techniques for estimating the parameters of stable distributions, only the FLOM, MOLC and QM yield to a closed-form expression for estimators of *all* four α-stable parameters [7,20].

For moment based methods like FLOM and MOLC, the selection of proper moment order is a critical bottleneck [21]. Furthermore, the higher order moments greater than α for α-stable distributions are infinite [22]. Although some proposed methods of matrix log-cumulants may lead to non-invertible equations and cause precarious performance. Moreover, the burden of these methods is too expensive in terms of time and number of computations or the variances of their estimates are high [23] . In order to overcome such difficulty, a Bayesian-based method to re-estimate the log-cumulants was proposed [24].

Through this paper, the general problem of α-stable distribution parameter estimation is analytically accomplished. Our approach is a Fourier domain scheme that computes the values of CF in $\omega = 1$ and ω_0 to hierarchical estimate of the four parameters of stable random variate. We derive an analytic formula for computation of ω_0 in general case. In addition to the theoretical results, we provide mean squared error (MSE), Kolmogorov–Smirnov (K–S) test and Kullback–Leibler (K–L) divergence to test the goodness-of-fit of the model.

Simulation results show that the proposed method outperforms the three main methods, i.e. FLOM, MOLC and QM for any value of the parameter α in the interval $(1, 2]$. Moreover, our scheme has less computation time compared to the FLOM and QM schemes, whereas the MOLC shows low accuracy. The problem of error propagation is also discussed.

Simulation results depict that our method is more efficient in terms of accuracy and simplicity. The proposed approach is validated using three different types of SAR signals and experimental results show that this new method can estimate the parameters and model the PDF of SAR signals more precisely.

The remaining of this paper is organized as follows. Section 2 is devoted to the α-stable distributions and related parameter estimation methods. In Section 3, we present our novel CF based method for estimating the four parameters of α-stable distributions. Section 4 includes the quantitative appraisal of proposed approach as well as some of the three main existing estimators with closed-form expression through the simulations. A comparison with ML estimator, as a reference of numerical methods is also provided. Experimental validations and performances are presented in Section 5 for real SAR return signals. Finally, the paper is concluded in Section 6.

2. α-stable distribution

In this section, we peruse the α-stable distribution and its CF. Suppose that $-\infty < x < \infty$ and x is distributed according to a stable law, i.e. $x \sim S(\alpha, \beta, \gamma, \delta)$, so its PDF, $f(x; \alpha, \beta, \gamma, \delta)$, is completely determined by four parameters [25]; α is the *characteristic exponent* and it determines the *shape* of the distribution, $(0 < \alpha \le 2)$, β is the index of *skewness*, $(-1 \le \beta \le 1)$, γ is the *dispersion* or *scale* parameter of the distribution and plays a similar role to the variance of the Gaussian distribution, $(\gamma > 0)$, and δ is the *location* parameter, $(\delta \in \mathbb{R})$. The case $\beta = 0$ which is correspond to the symmetric α-stable distribution [26]. α-stable distribution is usually provided by taking the inverse Fourier transform of its CF; however, a closed-form formula does not exist for its density function [7]. The CF of the α-stable distribution is defined as the following [25],

$$\varphi(\omega;\, \alpha, \beta, \gamma, \delta) = E\{e^{j\omega x}\} = \begin{cases} \exp\left\{-\gamma|\omega|^{\alpha}\left[1 - j\beta \mathrm{sign}(\omega)\tan\left(\frac{\pi\alpha}{2}\right)\right] + j\delta\omega\right\}, & \alpha \neq 1 \\ \exp\left\{-\gamma|\omega|\left[1 + j\beta \mathrm{sign}(\omega)\frac{2}{\pi}\ln|\omega|\right] + j\delta\omega\right\}, & \alpha = 1, \end{cases} \tag{1}$$

where $\mathrm{sign}(\cdot)$ is the sign function. From the above equation, it is found that the expression for α-stable CF has a discontinuity at $\alpha = 1$. Note that when $\alpha = 2$, the four parameters are well identified except the parameter β. If β is positive (negative) the distribution is skewed to the right (left) and this affects particularly the tails of the distribution. Analytical formula for PDF expression only exists for two special values of α for $1 \le \alpha \le 2$. The case $\alpha = 2$,

$$f_G(x) = f(x|\alpha = 2, \beta, \gamma, \delta) = \frac{1}{2\sqrt{\pi\gamma}} e^{-\frac{(x-\delta)^2}{4\gamma}}, \tag{2}$$

which is basically the classical Gaussian distribution. The other special case is $\alpha = 1, \beta = 0$,

$$f_C(x) = f(x|\alpha = 1, \gamma, \delta) = \frac{\gamma}{\pi[(x-\delta)^2 + \gamma^2]}, \tag{3}$$

corresponds to the Cauchy distribution and also is called Lorentz distribution in physics. In this study, we focus on α-stable distributions with $1 < \alpha \le 2$. In the following, we briefly review the three main methods used in analytical parameter estimation of α-stable distribution.

2.1. Parameter estimation of α-stable distribution

2.1.1. FLOM

Because of the thick tails, stable distributions have infinite second (or higher) order moments, except for the case of $\alpha = 2$ (Gaussian distribution). Therefore, the classical estimation methods based on the integer moments order are unenforceable [21]. Expression for the FLOM of α-stable distribution has previously been given by Nikias and Shao [25]. Kuruoglu [27] developed the theory of FLOM to general α-stable distributions. Based on this method, the characteristic exponent estimation is given as the solution to the following equation; although, the argument of sinc function in [27] is wrong and π must be omitted.

$$\mathrm{sinc}\left(\frac{p}{\hat{\alpha}}\right) = \left[q\left(\frac{A_p A_{-p}}{\tan q} + S_p S_{-p}\tan q\right)\right]^{-1}, \tag{4}$$

where

$$A_p = E\{|z|^p\}, \quad S_p = E\{z^{<p>}\}, \tag{5}$$

are the absolute and the signed FLOMs and $z^{<p>} = \mathrm{sign}(z)|z|^p$. $0 < p < \alpha$ is the moment order and is not necessarily an integer number, $q = \frac{p\pi}{2}$. In FLOM method, various estimators for α, β and γ are provided but in all cases the location parameter is assumed zero (zero-mean α-stable distribution). Since we assume $1 < \alpha \le 2$, the location parameter is equal to the mean of distribution [28]. To apply the FLOM method to general α-