Signal Processing 129 (2016) 1–11

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Rank constrained nuclear norm minimization with application to image denoising

Xixi Jia, Xiangchu Feng*, Weiwei Wang

School of mathematics and statistics, XiDian University, Xi'an 710126, China

ARTICLE INFO

Article history:
Received 1 December 2015
Received in revised form
12 April 2016
Accepted 24 May 2016
Available online 26 May 2016

Keywords:
Low rank approximation
Non-convex optimization
Nuclear norm minimization

ABSTRACT

In the low rank matrix approximation problem, the well known nuclear norm minimization (NNM) problem plays a crucial role and attracts significant interests in recent years. In NNM the regularization parameter λ plays a decisive part, λ controls both the rank of the solution and the extent of the thresholding. However, it is hard for a single λ to balance the two issuses, and meanwhile the solving method calls singular value decomposition (SVD), of which the computational complexity is impracticable when the scale of the problem becomes large. This paper presents a *rank constrained nuclear norm* minimization (RNNM) method, in which the rank and the extent of thresholding are controlled separately by an added parameter k. More importantly, by proving its equivalence with an unconstrained bi-convex optimization problem RNNM can be solved in SVD free manner. In this paper, a SOR (Successive Over Relaxation) algorithm is designed for the equivalent bi-convex problem and its convergence is proved. We show that RNNM has a unique global optimal solution although being non-convex. We explicitly analyse the structure of the solution for the bi-convex problem and show some interesting properties. Finally, we verify the effectiveness of RNNM in image denoising. Experimental results show that the proposed solving method works faster than SVD based method. Thanks to the well balance of rank and thresholding, RNNM achieves superior results than the state-of-the-art methods in image denoising such as BM3D, SAIST in terms of both quantity measure and visual quality.

1. Introduction

Low rank approximation aims at approximating the underlying low rank matrix from its degraded one, and has been widely used in image processing, computer vision and pattern recognition. For instance, the texture of an image is believed to be of low rank due to its reduplication [27]. Therefore, low rank approximation can be used for cartoon texture decomposition. Meanwhile it can also be used in background foreground extraction [31,25] and multiple category classification [1]. It is observed that the matrix constructed by similar image patches is of low rank [18], thus low rank approximation can be used as an efficient tool for low level vision problems such as image denoising [18,11,17,34] and video denoising [19]. In recent years, plenty of works have been done on the theorem and algorithm of low rank approximation [28,15,16,4,21,13].

The original low rank approximation formulated as follows is a non-convex problem:

$$\min_{\boldsymbol{X}} \ \operatorname{rank}(\boldsymbol{X}) \quad \text{s. t.} \ \ \|\boldsymbol{X}-\boldsymbol{Y}\|_F^2 \le \epsilon \tag{1}$$

It is not easy to handle (1). To make the problem practicable, a large amount of works have been done on discussing the following convex relaxation problem:

$$\min_{\boldsymbol{X}} \ \|X\|_* \quad \text{s. t.} \ \|\boldsymbol{X}-\boldsymbol{Y}\|_F^2 \le \epsilon \tag{2}$$

It has been proved that solving the convex nuclear norm minimization problem leads to a near optimal low-rank solution [7]. By casting the constrained optimization into an unconstrained one, (2) is equivalent to problem (3):

$$\min_{\boldsymbol{X}} \frac{1}{2}\|\boldsymbol{Y}-\boldsymbol{X}\|_F^2 + \lambda\|X\|_* \tag{3}$$

where λ is a positive constant. In the early stage, [16] proved that (3) can be solved using SDP(semi-definite programming), but it is only effective in solving problems of small scale. Recently, [4] proved that problem (3) can be solved easily by singular value thresholding. That is, the optimal solution of (3) can be obtained by $\boldsymbol{X}^* = \boldsymbol{U}\mathcal{S}_\lambda(\Sigma)\boldsymbol{V}^T$, where $\boldsymbol{Y} = \boldsymbol{U}\Sigma\boldsymbol{V}^T$ is the SVD of $\boldsymbol{Y}$ and $\mathcal{S}_\lambda(\Sigma)$ is

* Corresponding author.
E-mail address: xcfeng@mail.xidian.edu.cn (X. Feng).

http://dx.doi.org/10.1016/j.sigpro.2016.05.026
0165-1684/

the soft-thresholding function. The work of [4] provided closed form solution of problem (3), since then nuclear norm minimization (NNM) shows its power in low rank approximation and has been widely used in matrix completion [6] and low rank representation [20]. Albeit its success, for NNM there still has certain limitations .

For NNM the regularization parameter λ plays a decisive part, which means it controls both the rank of the solution and the extent of thresholding on singular values. However, in many applications (such as image denoising) it is hard for a single λ to balance the two issues. Large λ lowers the rank of the solution but over-penalties the large singular values, small λ preserves the large singular values but increases the rank. In recent years a great quantity of works have been done on discussing the balance of rank and thresholding, therefore to avoid the over-penalty. Zhang et al. [32] proposed a Truncated Nuclear Norm Regularization (TNNR) method, in which it keeps the large singular values and thresholds on some specific small singular values. TNNR is not flexible enough since it makes a binary decision on whether to regularize a specific singular value or not. Wen. et al. proposed a low rank factorization model [30] which restricts the rank of the solution by a dynamically rectified parameter but with no regularization on large singular values. In real applications such as image denoising the large singular values are also contaminated by noise, it is unreasonable to keep the large singular values untouched. Gu et al. [18] proposed a weighted nuclear norm minimization method (WNNM) to balance the rank and thresholding, and achieves competitive results, but it is not easy to estimate an appropriate weight for WNNM. Also large amounts of non-convex method was proposed in [22].

Apart from the over-penalty problem, the solving method of all these singular value regularized methods call SVD operation, and the computational complexity of SVD is impracticable when the problem scale becomes large. To reduce the computational complexity, Cai [5] designed fast singular value thresholding method using the dual of SVT. Since in their work, the input matrix should be preprocessed by complete orthogonal decomposition which requires $O(mn, \min(m, n))$, so the reduction of computational complexity is still limited. The work of [30] using matrix factorization to approximate low rank. In other thread of works, Mu et al. [25] proposed a compressed optimization by random projection. Ma et al. [23] using a linear-time approximate SVD [14]. However, these methods are unstable. Although these works avoid SVD, they cannot solve the over-penalty problem. To the best of our knowledge, there are few works have been done on solving the over-penalty problem and avoiding SVD simultaneously.

On account of this, in this paper, we propose a rank constrained nuclear norm minimization (RNNM) method. In RNNM, the rank and the extent of thresholding are controlled separately by an added parameter k. k restricts the rank and λ controls the thresholding. Benefiting from the introduced parameter k, RNNM well balances the rank and thresholding, refrains from over-penalty of NNM. At the same time, we prove that RNNM is equivalent to a bi-convex matrix factorization problem, which can be solved in SVD free manner. We prove that, RNNM and its equivalent bi-convex matrix factorization form, although being non-convex, can guarantee a global optimal solution. In addition, the SOR (successive over relaxed) algorithm is designed for the bi-convex form RNNM problem, and its convergence analysis is presented. We show that the solution of the equivalent bi-convex form RNNM model has some nice properties, which may be profitable for some specific problems. To test the effectiveness of RNNM, we apply RNNM for image denoising, by utilizing the nonlocal self-similarities priori of images, the RNNM model achieves satisfactory denoising results. Meanwhile, we compared the computational time of RNNM and SVD based low rank models. The experimental results verified the effectiveness of RNNM, that is RNNM indeed solve the over-penalty problem and meanwhile save computational time.

The contribution of this paper is three folds.

- We propose the RNNM model to solve the over-penalty problem of standard NNM.
- A SVD free algorithm is designed for RNNM, and convergence analysis is given.
- We apply RNNM for image denoising, and meanwhile we verify the computational efficiency of RNNM.

The rest of this paper is organized as follows: in Section 2, we exhibit our RNNM model and prove that it is equivalent to a bi-convex factorization model. Meanwhile, we prove that the bi-convex factorization model can be solved in SVD free manner. Furthermore, we give specific analysis of the bi-convex form RNNM and show some useful properties of its solution. In Section 3, we present a SOR algorithm for bi-convex form RNNM and prove its convergence. In Section 4, we adopt the proposed RNNM for image denoising to demonstrate the effectiveness of our method. In Section 5, we give experimental results to show that RNNM works faster than SVD based low rank approximation, and also show that RNNM performs excellent in image denoising. Conclusions are given in Section 6.

Notations: For a matrix $P \in \mathbb{R}^{m\times n}$, we assume $m > n$. Let $P_{i\cdot}$ and $P_{\cdot j}$ denote the ith row and the jth column of $\boldsymbol{P}$, $\|P\|_F$ the Frobenius norm and $\|P\|_*$ the nuclear norm. $\|\cdot\|$ denotes Euclidean norm of a vector and $\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors with matching dimension. Denote by r the rank of the data matrix $\boldsymbol{Y}$.

2. Rank constrained nuclear norm minimization

As stated in Section 1, NNM uses a single λ to balance the rank and threshold. Soft-thresholding uses a large λ results in a low rank solution, meanwhile it penalizes on large singular values too much. On contrary, soft-thresholding uses small λ to preserve the large singular values; however it cannot come at a low rank solution. To settle this problem, we introduce a new parameter k in the standard NNM and cast the problem to a constrained non-convex optimization problem RNNM as

$$\min_{\boldsymbol{X}} \frac{1}{2}\|\boldsymbol{Y}-\boldsymbol{X}\|_F^2 + \lambda\|\boldsymbol{X}\|_* \\ \text{s. t. } \operatorname{rank}(\boldsymbol{X}) \le k \tag{4}$$

Generally, the parameter k is smaller than the rank of $\boldsymbol{Y}$.

Problem (4) is non-convex constrained optimization problem which is absolutely different to the convex unconstrained NNM (3). We give a theorem to show that RNNM has unique global optimal solution-the rank restricted soft thresholding (RRST) as shown in Fig. 1.

Theorem 1. *$\forall\, \boldsymbol{Y} \in \mathbb{R}^{m\times n}$, denote by $\boldsymbol{Y} = \boldsymbol{U}\Sigma\boldsymbol{V}^T$ the SVD of it. For RNNM in (4) its solution $\widehat{\boldsymbol{X}}$ is unique and can be written as $\widehat{\boldsymbol{X}} = \boldsymbol{U}\mathcal{S}_{k,\lambda}(\Sigma)\boldsymbol{V}^T$, where $\mathcal{S}_{k,\lambda}(\Sigma)$ is a rank restricted soft-thresholding (RRST) operator*

$$S_{k,\lambda}(\Sigma)_{ii} = \begin{cases} \max(\Sigma_{ii}-\lambda, 0), & i = 1\cdots k \\ 0, & o.\ w \end{cases}$$

Therefore, the rank and extent of thresholding are treated separately. Fig. 1 gives an illustration of the RRST operator as formulated in Theorem 1, and makes comparison with the traditional soft-thresholding and hard-thresholding operators. In Fig. 1 (left),