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Frequency estimation of sinusoids from nonuniform samples

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ABSTRACT

Sinusoid signals with multiple frequencies appear in various systems and their frequencies may carry some important features. Frequency estimation from their discrete samples is one of the fundamental problems and many frequency estimators have been proposed for uniform sampling setting. In this paper, frequency estimators based on adaptive notch filtering are proposed for nonuniform sampling setting. We observe that some dynamic systems associated with adaptive notch filters can be solved in nonuniformly sampled time steps with high accuracy. This leads us to propose a digital adaptive notch filtering method to estimate frequency of a sinusoidal signal with single frequency from its nonuniform samples. The proposed method exhibits convergent and robust frequency estimation in the presence of random sampling noises, and its variance is comparable to the Cramer–Rao lower bound in the presence of additive white noise. The above method designed for single frequency estimation could track abrupt single frequency change of an input signal, but it is not applicable directly for multiple frequency estimation. Our simulations show that the proposed estimators have robust performance for sinusoidal signals with multiple distinct frequencies, and they can be used to separate two very close frequencies of an input signal in a highly noisy sampling environment.

1. Introduction

Consider a mixture of sinusoidal signals, whose kth component has amplitude A_k, frequency θ_k and phase ϕ_k, $1 \le k \le K$,

$$y(t) = \sum_{k=1}^{K} A_k \sin(\theta_k t + \phi_k). \tag{1.1}$$

Such sinusoidal signals are encountered in active noise and vibration control, wireless communications, audio, radar and sonar signal processing [1–4]. In telecommunication systems, the frequencies θ_k, $1 \le k \le K$, contain carrier's phase information necessary for synchronization of demodulators or other components of a receiver system.

The estimation problem of frequencies θ_k, $1 \le k \le K$, of the signal y is a fundamental problem in systems theory with many applications. It has been intensively studied in signal processing, instrumentation and measurements, and control theory. Many frequency estimators have been proposed, including adaptive notch filtering, time frequency representation, phase locked loop, eigensubspace tracking, extended Kalman filtering, internal model method, etc., see [5–8] and references therein.

Most of existing estimators are derived for uniformly sampled data $y(n\Delta T)$, $n \ge 0$, with uniform sampling frequency $1/\Delta T$, and often only for a single frequency, i.e., $K=1$. In this paper, we consider multiple frequency estimation problem of the signal y from its nonuniform samples,

$$z_n = y(T_n) + w_n, \quad n \ge 0, \tag{1.2}$$

corrupted by additive noises w_n, where T_n, $n \ge 0$, are sampling times.

Nonuniform sampling arises in many applications, such as computer graphics, frequency scanning interferometry, magnetic resonance imaging, computer tomography scans and Radon imaging [9–14]. Uniform sampling is well studied and it has been widely used in engineering applications. However in some applications, nonuniform sampling is necessary and it has better performance. For instance, in antialiasing in computer graphics [15], better results can be obtained with random sampling instead of uniform sampling. The sampling operation could be costly, and a low number of samples (but not necessarily uniform) is more desirable. For instance, in frequency scanning interferometry, the sampling effort is measured by the acquisition time at a given point, and the optimal sampling scheme is usually nonuniform [16]. Nonuniformly sampled data are harder to analyze and the related methods, even for a "simple" task of obtaining the discrete nonuniform Fourier transform, are much more difficult and often

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iterative [9,17]. Several statistical frequency estimators (based on maximum likelihood estimation and filter banks) from nonuniformly sampled data have been proposed in the literature [11,18–23]. For nonuniform sampling problems in signal processing, the reader may refer to [24–26].

In many applications of signal processing, it is desirable to eliminate or extract sine waves from observed data or to estimate their unknown frequencies. Since the frequencies often vary with time, it is useful to apply adaptive notch filters (ANFs) that adapt their notch frequencies as a function of the observed time series, see [27,28] and references therein. The ANF method is one of the most suitable techniques to separate sinusoidal components of unknown frequencies buried in noise, and/or retrieve such periodic components [29–35]. It is robust in the presence of sampling noise and it is capable of changing the notch frequency accordingly. Various architectures have been proposed for the construction of adaptive notch filters, see for instance [1,36–41].

Frequency estimation problem using ANF is modeled as a nonlinear system identification of a dynamic system either in continuous time (CT) (e.g. [1]) or in discrete time (DT) (e.g. [29]). The CT model systems are native to the physical world, they have a built-in capability to cope with the nonuniformly sampled signal, and they offer certain advantages over purely DT model systems [45,46]. Compared to the DT model, direct estimation of CT models is usually stable, accurate and free from undesirable sensitivity problems, particularly at high sampling rates. The frequency estimator developed in this paper is based on ANFs, which are governed by some CT dynamic systems [29–35,42–44]. We mainly focus on a particular ANF governed by the following dynamic system:

$$\begin{cases} Dx_1 = x_2 \\ Dx_2 = -2\xi\theta x_2 - \theta^2 x_1 + \theta^2 y \\ D\theta = -\gamma(\theta^2 y - 2\xi\theta x_2)x_1, \end{cases} \tag{1.3}$$

where D represents the derivative with respect to time t, x_1, x_2, θ are states of the system, y is the excitation sinusoidal input with single frequency θ_0 (i.e., $K=1$ in (1.1)), ξ is the notch depth, and γ is the adaptation speed. The above system of nonlinear ordinary differential equations (ODE) has good noise rejection capability. It was proposed by Regalia in [1] as a DT filter, it was later adapted by Bodson and Douglas [42] for a CT system, and finally a modified version was proposed by Hsu et al. [29]. We have chosen this dynamic system due to its superior performance compared to the other systems that can be used with a similar discretization procedure, see Section 2.5 for performance comparison.

The main difficulty in handling CT dynamic systems directly is the problem of evaluating derivatives of the input signal (with unknown parameters) from its nonuniform samples numerically, the numerical differentiation process in a highly noisy environment is usually unstable and impractical [47–49]. In this paper, we propose a Taylor-like approximation of the dynamic system (1.3) that is robust to noise and achieves high accuracy. Based on the above approximation, we introduce an ANF method (2.13) to estimate the frequency of a sinusoidal signal from its nonuniform samples. The proposed discrete ANF method reconciles the merits of CT models while restricting itself to operate directly on the DT data.

This paper is organized as follows. Section 2 discusses a Taylor-like approximation technique to solve the system (1.3) and a frequency estimation of the unknown input signal with single frequency. We propose the frequency estimator (2.13), perform the local stability analysis, and study its convergence, noise characteristics, statistical properties, and comparison to the conventional ODE solver for the dynamic systems associated with the ANF methods. We also perform a comparison of our method with a state of the art discrete ANF [40]. Section 3 describes extensions of the single frequency estimator (2.13) to multiple frequency estimations with two configurations, the cascade ANF method and the prefiltering ANF method. The two proposed multiple frequency estimators have robust performance for sinusoidal signals with multiple distinct frequencies or related harmonic frequencies. Most of the known frequency estimators have poor performance when input signal has two very close frequencies in a highly noisy environment, a pathological case where the estimation error is related to both the difference in the frequency and the noise level, see [50] and references therein. Our simulation indicates that the cascade ANF method has sound performance even in the separation of very close frequencies of the input signal in a highly noisy sampling environment. We close the paper with concluding remarks in Section 4.

Notation: We use Euler notation for expressing derivatives, D^n instead of D_t^n to denote the nth derivative with respect to time t.

2. Single frequency estimation

Consider a sinusoidal input,

$$y(t) = A\sin(\theta_0 t + \phi), \tag{2.1}$$

where A, θ_0, ϕ are its amplitude, frequency, and phase respectively. In the first subsection, we propose a discrete ANF method to estimate frequency θ_0 of the sinusoid signal y from its nonuniform samples,

$$z_n = y(T_n) + w_n, \quad n \ge 0, \tag{2.2}$$

corrupted by additive noise w_n at sampling times T_n, $n \ge 0$. Then in the next four subsections, we discuss local stability, convergence, approximation error, statistical characterization, and extensions of the proposed ANF method. We also compare the performance of our approach (2.13) with some of the existing ANF methods for estimating frequency in Section 2.2 and 2.5.

2.1. The proposed method

The dynamical system (1.3) converges to its unique periodic orbit,

$$[x_1,\ x_2,\ \theta]^T = \left[\frac{-A\cos(\theta_0 t + \phi)}{2\xi},\ \frac{A\theta_0\sin(\theta_0 t + \phi)}{2\xi},\ \theta_0\right]^T, \tag{2.3}$$

when the adaption speed γ satisfies

$$0 < \gamma < 4\xi/A^2 \tag{2.4}$$

[29] . The dynamical system (1.3) can be rewritten as follows:

$$D\mathbf{X} = \mathbf{F}(t, \mathbf{X}), \tag{2.5}$$

where $\mathbf{X} = [x_1, x_2, \theta]^T$ is the state of the system, and

$$\mathbf{F}(t, \mathbf{X}) = \left[x_2,\ -2\xi\theta x_2 - \theta^2 x_1 + \theta^2 y,\ \gamma(2\xi\theta x_2 - \theta^2 y)x_1\right]^T$$

is a real analytic function of t and $\mathbf{X}$. Therefore $\mathbf{X}(t)$ is real analytic by the Cauchy–Kovalevskaya theorem [51, Theorem 2 of Chapter 4], and it has the following Taylor expansion:

$$\mathbf{X}(t) = \sum_{k=0}^{m}\frac{D^k\mathbf{X}(T_n)}{k!}t^k + \int_{T_n}^{t} D^m\mathbf{F}(s, \mathbf{X}(s))\frac{(t-s)^m}{m!}ds \tag{2.6}$$

for all $T_n \le t \le T_{n+1}$ and $m \ge 0$.

For an input signal of sinusoidal type, we observe that the state vector $\mathbf{X}(t)$ of the dynamical system (1.3) can be approximated by Taylor polynomials $\sum_{k=0}^{m} D^k\mathbf{X}(T_n)(t-T_n)^k/k!$ of low order $m \le 4$,