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Survey paper

Synchronization in complex networks of phase oscillators: A survey[☆]

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ABSTRACT

The emergence of synchronization in a network of coupled oscillators is a fascinating subject of multidisciplinary research. This survey reviews the vast literature on the theory and the applications of complex oscillator networks. We focus on phase oscillator models that are widespread in real-world synchronization phenomena, that generalize the celebrated Kuramoto model, and that feature a rich phenomenology. We review the history and the countless applications of this model throughout science and engineering. We justify the importance of the widespread coupled oscillator model as a locally canonical model and describe some selected applications relevant to control scientists, including vehicle coordination, electric power networks, and clock synchronization. We introduce the reader to several synchronization notions and performance estimates. We propose analysis approaches to phase and frequency synchronization, phase balancing, pattern formation, and partial synchronization. We present the sharpest known results about synchronization in networks of homogeneous and heterogeneous oscillators, with complete or sparse interconnection topologies, and in finite-dimensional and infinite-dimensional settings. We conclude by summarizing the limitations of existing analysis methods and by highlighting some directions for future research.

1. Introduction

Synchronization in networks of coupled oscillators is a pervasive topic in various scientific disciplines ranging from biology, physics, and chemistry to social networks and technological applications. A coupled oscillator network is characterized by a population of heterogeneous oscillators and a graph describing the interaction among the oscillators. These two ingredients give rise to a rich dynamic behavior that keeps on fascinating the scientific community.

Within the rich modeling phenomenology on synchronization among coupled oscillators, this paper focuses on the widely adapted model of a continuous-time and periodic limit-cycle oscillator network with continuous, bidirectional, and anti-symmetric coupling. We consider a system of n oscillators, each characterized by a phase angle $\theta_i \in \mathbb{S}^1$ and a natural rotation frequency $\omega_i \in \mathbb{R}$. The dynamics of each isolated oscillator are thus $\dot{\theta}_i = \omega_i$ for $i \in \{1, \ldots, n\}$. The interaction topology and coupling strength among the oscillators are modeled by a connected, undirected, and weighted graph $G(\mathcal{V}, \mathcal{E}, A)$ with nodes $\mathcal{V} = \{1, \ldots, n\}$, edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and positive weights $a_{ij} = a_{ji} > 0$ for each undirected edge $\{i, j\} \in \mathcal{E}$. The interaction between neighboring oscillators is assumed to be additive, anti-symmetric, diffusive,[2] and proportional to the coupling strengths a_{ij}. In this case, the simplest 2π-periodic interaction function between neighboring oscillators $\{i, j\} \in \mathcal{E}$ is $a_{ij} \sin(\theta_i - \theta_j)$, and the overall model of coupled phase oscillators reads as

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \quad i \in \{1, \ldots, n\}. \tag{1}$$

Despite its apparent simplicity, this coupled oscillator model gives rise to rich dynamic behavior, and it is encountered in many

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2 The interaction between two oscillators is *diffusive* if its strength depends on the corresponding phase difference; such interactions arise for example in the discretization of the Laplace operator in diffusive partial differential equations.

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scientific disciplines ranging from natural and life sciences to engineering. This paper surveys recent results and applications of the coupled oscillator model (1) and of its variations.

The motivations for this survey are manifold. Recent years have witnessed much theoretical progress and novel applications, which are not covered in existing surveys (Acebrón, Bonilla, Vicente, Ritort, & Spigler, 2005; Arenas, Díaz-Guilera, Kurths, Moreno, & Zhou, 2008; Dorogovtsev, Goltsev, & Mendes, 2008; Strogatz, 2000) published in the physics literature. Indeed, control scientists have shown an increasing interest in complex networks of coupled oscillators and have recently contributed many novel approaches and results. Much of this interest has focused on (i) synchronization rather than more complex dynamic phenomena, (ii) finite numbers of oscillators with a non-trivial interaction topology, and (iii) connections with graph theory and multi-agent systems. It is therefore timely to provide a comprehensive review in a unified control-theoretical language of the best known results in this area. With this aim, this survey provides a systems and control perspective to coupled oscillator networks, focusing on quantitative results and control-relevant applications in sciences and technology.

1.1. Mechanical analog and basic phenomenology

A mechanical analog of the coupled oscillator model (1) is the spring network shown in Fig. 1. This network consists of a group of particles constrained to move on a unit circle and assumed to move without colliding. Each particle is characterized by its angle $\theta_i \in \mathbb{S}^1$ and frequency $\dot{\theta}_i \in \mathbb{R}$, and its inertial and damping coefficients are $M_i > 0$ and $D_i > 0$ respectively. Pairs of interacting particles i and j are coupled through a linear-elastic spring with stiffness $k_{ij} > 0$. The external forces and torques acting on each particle are a viscous damping force $D_i\dot{\theta}_i$ opposing the direction of motion, an external driving torque $\tau_i \in \mathbb{R}$, and an elastic restoring torque $k_{ij}\sin(\theta_i - \theta_j)$ between pairs of interacting particles. The overall spring network is modeled by a graph, whose nodes are the particles, whose edges are the linear-elastic springs, and whose edge weights are the positive stiffness coefficients $k_{ij} = k_{ji}$. Under these assumptions, it can be shown (Dörfler, Chertkov, & Bullo, 2013) that the system of spring-interconnected particles obeys the dynamics

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = \tau_i - \sum_{j=1}^{n} k_{ij}\sin(\theta_i - \theta_j), \quad i \in \{1, \ldots, n\}. \tag{2}$$

In the limit of small masses M_i and uniformly-high viscous damping $D = D_i$, that is, $M_i/D \approx 0$, we recover the coupled oscillator dynamics (1) from its mechanical analog (2) with natural rotation frequencies $\omega_i = \tau_i/D$ and with coupling strengths $a_{ij} = k_{ij}/D$.

The mechanical analog in Fig. 1 illustrates the basic phenomenology displayed by the oscillator network (1). The spring-interconnected particles are subject to a competition between the external driving forces ω_i and the internal restoring torques $a_{ij}\sin(\theta_i - \theta_j)$. Hence, the interesting coupled oscillator dynamics (1) arise from a trade-off between each oscillator's tendency to align with its natural frequency ω_i and the synchronization-enforcing coupling $a_{ij}\sin(\theta_i - \theta_j)$ with its neighbors. Intuitively, a weakly coupled and strongly heterogeneous (i.e., with strongly dissimilar natural frequencies) network does not display any coherent behavior, whereas a strongly coupled and sufficiently homogeneous network is amenable to synchronization, where all frequencies $\dot{\theta}_i(t)$ or even all phases $\theta_i(t)$ become aligned.

1.2. History, related applications, and theoretical developments

A brief history of synchronization: The scientific interest in synchronization of coupled oscillators can be traced back to the work by Huygens (1893) on "an odd kind of sympathy" between coupled pendulum clocks, mutual influence of organ pipes (Rayleigh, 1896), locking phenomena in circuits and radio technology (Adler, 1946; Appleton, 1922; Van Der Pol, 1927), the analysis of brain waves and self-organizing systems (Wiener, 1948, 1958), and it still fascinates the scientific community nowadays (Strogatz, 2003; Winfree, 2001). We refer to Blekhman (1988) and Pikovsky, Rosenblum, and Kurths (2003) for a detailed historical account of synchronization studies.

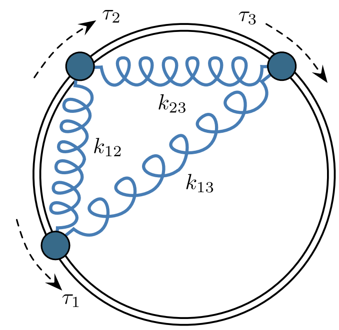

Fig. 1. Mechanical analog of a coupled oscillator network.

A variation of the considered coupled oscillator model (1) was first proposed by Winfree (1967). Winfree considered general (not necessarily sinusoidal) interactions among the oscillators. He discovered a phase transition from incoherent behavior with dispersed phases to synchrony with aligned frequencies and coherent (i.e., nearby) phases. Winfree found that this phase transition depends on the trade-off between the heterogeneity of the oscillator population and the strength of the mutual coupling, which he could formulate by parametric thresholds. However, Winfree's model was too general to be analytically tractable. Inspired by these works, Kuramoto (1975) simplified Winfree's model and arrived at the coupled oscillator dynamics (1) with a complete interaction graph and uniform weights $a_{ij} = K/n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j=1}^{n}\sin(\theta_i - \theta_j), \quad i \in \{1, \ldots, n\}. \tag{3}$$

In an ingenious analysis, Kuramoto (1975, 1984a) showed that synchronization occurs in the model (3) if the coupling gain K exceeds a certain threshold K_{critical} function of the distribution of the natural frequencies ω_i. The dynamics (3) are nowadays known as the *Kuramoto model* of coupled oscillators, and Kuramoto's original work initiated a broad stream of research. A compelling historical perspective is offered by Strogatz (2000). We also recommend the surveys by Acebrón et al. (2005), Arenas et al. (2008) and Dorogovtsev et al. (2008).

Canonical model and prototypical example: Diffusively-coupled phase oscillators appear to be quite specific at first glance, but they are locally *canonical models* for weakly coupled and periodic limit-cycle oscillators (Hoppensteadt & Izhikevich, 1997). This fact is established in work by the computational neuroscience community which has developed different approaches (Ermentrout & Kopell, 1984, 1991; Hoppensteadt & Izhikevich, 1997; Izhikevich, 2007; Izhikevich & Kuramoto, 2006) to reduce general periodic limit-cycle oscillators and weak interaction models to diffusively-coupled phase oscillator networks of the form

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{n} h_{ij}(\theta_i - \theta_j), \tag{4}$$

where $h_{ij} : \mathbb{S}^1 \to \mathbb{R}$ are 2π-periodic coupling functions. Among such phase oscillator networks, the often encountered and most