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On backward shift algorithm for estimating poles of systems[☆]

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ABSTRACT

In this paper, we present an algorithm for estimating poles of linear time-invariant systems by using the backward shift operator. We prove that poles of rational functions, including zeros and multiplicities, are solutions to an algebraic equation which can be obtained by taking backward shift operator to the shifted Cauchy kernels in the unit disc case. The algorithm is accordingly developed for frequency-domain identification. We also prove the robustness of this algorithm. Some illustrative examples are presented to show the efficiency in systems with distinguished and multiple poles.

1. Introduction

System identification is to build mathematical models which fit the measured data from discrete or continuous systems. A number of methods have been developed for this problem, such as Gu and Khargonekar (1992), Helmicki, Jacobson, and Nett (1991), Pintelon, Guillaume, Rolain, Schoukens, and Hamme (1994), Wahlberg (1991), Wahlberg (1994), Wahlberg and Mäkilä (1996). A classical guidebook for one getting to know this topic is Ljung (1999). For identification of linear time-invariant (LTI) systems, a priori-knowledge of poles is important, especially for the methods that adopt rational orthogonal bases such as in de Vries and Van den Hof (1998), Ninness (1996), Ninness and Gustafsson (1994) and Ninness, Hjalmarsson, and Gustafsson (1997). In these methods, the estimated poles are used to construct rational orthogonal basis functions. A collection of the excellent results is Heuberger, Van den Hof, and Wahlberg (2005).

In unit disc case, the general setting of a rational orthogonal basis is

$$\mathcal{B}_k(z) = \mathcal{B}_{\{a_1,\ldots,a_k\}}(z) \triangleq \frac{\sqrt{1-|a_k|^2}}{1-\bar{a}_k z}\prod_{l=1}^{k-1}\frac{z-a_l}{1-\bar{a}_l z}, \tag{1}$$

where a_ks $(k = 1, \ldots)$ are in the unit disc, ($\bar{a}$ means conjugation of a). Many researchers work on choosing optimal n-poles $\{a_k\}_{k=1}^{n}$ in order to define the best rational orthogonal bases for a system. Oliveira e Silva derived the optimal pole conditions for the Laguerre, Kautz and general orthogonal basis function models in Oliveira e Silva (1995a,b, 1997), respectively. In Mi and Qian (2010, 2012) and Mi, Qian, and Wan (2012), adaptive selection of poles is studied. Other attempts to estimate optimal pole positions of a Laguerre model are given in Casini, Garulli, and Vicino (2003) and Sabatini (2000). Generally speaking, the pole estimation of an LTI system, in practice, is not easy.

For a discrete LTI system which is causal and stable, let $\{x_k\}$, $\{y_k\}$ be the input and output signals, respectively. There is a relation between $\{x_k\}$ and $\{y_k\}$ as

$$y_k = \{x_k\} * \{h_k\} = \sum_{l=0}^{+\infty} h_l x_{k-l}, \tag{2}$$

where $\{h_k\}$ is the impulse response. With an operator q, $qx(k) = x(k+1)$, is drawn into, (2) can be represented as

$$y_k = \sum_{l=0}^{+\infty} h_l x_{k-l} = \left(\sum_{l=0}^{+\infty} h_l q^{-l}\right) x_k. \tag{3}$$

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The related function

$$G(z)=\sum_{l=0}^{+\infty}h_l z^{-l} \tag{4}$$

is the transfer function of the system. The values of the transfer function for z on the unit circle are called frequency responses. Under the stability and causality assumption, $G(z)$ is a proper rational function with real coefficients. To identify $G(z)$ with general orthogonal bases, estimating poles for the basis functions plays a significant role.

As is well-known, the basis (1) can be obtained by the shifted Cauchy kernels $e_a(z)$ with the Gram–Schmidt process, where $e_a(z)$ is given by

$$e_a(z)=\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}.$$

For these kernels, there is a very good property when taking backward shift operator on them. Based on this property, we can estimate poles of e_a's instead of the orthogonal cases. In this paper, we are to locate poles of an LTI system based on a set of frequency domain measurements by using backward shift operator, which results in an algorithm, we call it *backward shift algorithm*.

This paper is arranged as follows. In Section 2, we study each case of taking backward shift operator to rational functions. After that, we introduce the backward shift algorithm in detail in Section 3. Examples are given in Section 4 to illustrate the proposed idea. Some conclusions are drawn in Section 5.

2. Backward shift on rational functions

2.1. Backward shift operator

The backward shift operator, denoted by $\mathbf{S}$,

$$\mathbf{S}(f)(z)=\frac{f(z)-f(0)}{z}, \tag{5}$$

is the Banach space adjoint of the forward shift operator $\mathbf{F}(f)(z)=zf(z)$ in the Hardy-2 space in the unit disc, viz.,

$$\langle \mathbf{S}(f),g\rangle=\langle f,\mathbf{F}(g)\rangle,\quad f,g\in H_2. \tag{6}$$

It is an important and interesting operator. Comprehensive studies in the operator and related topics can be found, for instance, in Aleksandrov (1979), Cima and Ross (2000) and Nikol'skiĭ (1986). It is well known that a collection of countably many reproducing kernels of the Hardy space H_2, viz., conjugates of the shifted Cauchy kernels, generates a backward shift invariant subspace.

For $0\neq a\in\mathbb{D}$, the unit disc, we notice the kernel $e_a(z)=\frac{1}{1-\bar{a}z}$ (for convenience we will call $\bar{a}$ a pole of it, although we know precisely it is $\frac{1}{\bar{a}}$) is an eigenvector of $\mathbf{S}$, viz.,

$$\mathbf{S}(e_a)(z)=\frac{e_a(z)-e_a(0)}{z}=\frac{\bar{a}}{1-\bar{a}z}.$$

Therefore,

$$\mathbf{S}^2(e_a)(z)=\mathbf{S}(\mathbf{S}(e_a))(z)=\frac{\bar{a}^2}{1-\bar{a}z},$$

and, in general,

$$\mathbf{S}^n(e_a)(z)=\frac{\bar{a}^n}{1-\bar{a}z}. \tag{7}$$

An n-tuple $(a_1,\ldots,a_n)$ in the unit disc corresponds to one of the following two n-tuples of partial fractions, being determined on whether some a_k's are zero. Denote by $b_1,\ldots,b_m$ all the distinguished ones among $a_1,\ldots,a_n$.

Case 1. If none of the distinguished b_k's is zero, then it corresponds to

$$\frac{1}{1-\bar{b}_1z},\ldots,\frac{1}{(1-\bar{b}_1z)^{l_1}},\ldots,\frac{1}{1-\bar{b}_mz},\ldots,\frac{1}{(1-\bar{b}_mz)^{l_m}},$$

where $l_1,\ldots,l_m$ are multiples of $b_1,\ldots,b_m$, respectively and $l_1+\cdots+l_m=n$.

A rational function p/q, where p and q are co-prime polynomials, is a non-degenerate linear combination of the above linearly independent set of functions if and only if the degree of q is equal to n, and the degree of p is less than n.

Case 2. If one of the distinguished b_k's is zero, say, $b_1=0$, with multiplicity l_1, then it corresponds to

$$1,\ldots,z^{l_1},\frac{1}{1-\bar{b}_2z},\ldots,\frac{1}{(1-\bar{b}_2z)^{l_2}},\ldots,\frac{1}{(1-\bar{b}_mz)^{l_m}},$$

where $l_1+\cdots+l_m=n$.

A rational function p/q, where p and q are co-prime polynomials, is a non-degenerate linear combination of the above linearly independent set of functions if and only if the degree of q is equal to $n-l_1$, and the degree of p is less than n.

These cases will be studied in detail in the following three subsections.

2.2. The distinguished non-zero poles case

In this subsection we treat the case where all b_k, $k=1,\ldots,n$, are different from each other, that is, each multiplicity is 1. Assume that f is of the form

$$f(z)=\sum_{k=1}^{n}\frac{\lambda_k}{1-\bar{b}_kz}, \tag{8}$$

where λ_ks are non-zero. Applying, consecutively, the backward shift operator $\mathbf{S}$ to $f(z)$ n times, we have

$$\begin{cases}\mathbf{S}(f)(z)=\dfrac{\lambda_1\bar{b}_1}{1-\bar{b}_1z}+\dfrac{\lambda_2\bar{b}_2}{1-\bar{b}_2z}+\cdots+\dfrac{\lambda_n\bar{b}_n}{1-\bar{b}_nz},\\ \mathbf{S}^2(f)(z)=\dfrac{\lambda_1\bar{b}_1^2}{1-\bar{b}_1z}+\dfrac{\lambda_2\bar{b}_2^2}{1-\bar{b}_2z}+\cdots+\dfrac{\lambda_n\bar{b}_n^2}{1-\bar{b}_nz},\\ \vdots\\ \mathbf{S}^n(f)(z)=\dfrac{\lambda_1\bar{b}_1^n}{1-\bar{b}_1z}+\dfrac{\lambda_2\bar{b}_2^n}{1-\bar{b}_2z}+\cdots+\dfrac{\lambda_n\bar{b}_n^n}{1-\bar{b}_nz}.\end{cases}$$

Since the b_ks are distinguished, $\{\frac{1}{1-\bar{b}_kz}\}_{k=1}^n$ is a linearly independent collection. There exists a unique non-zero sequence $\{\mu_k\}_{k=0}^n$ such that

$$\mu_0f(z)+\mu_1\mathbf{S}(f)(z)+\cdots+\mu_n\mathbf{S}^n(f)(z)=0. \tag{9}$$

Precisely,

$$\begin{cases}0=(\mu_0+\mu_1\bar{b}_1+\cdots+\mu_{n-1}\bar{b}_1^{n-1}+\mu_n\bar{b}_1^n)\dfrac{\lambda_1}{1-\bar{b}_1z}\\ \quad+(\mu_0+\mu_1\bar{b}_2+\cdots+\mu_{n-1}\bar{b}_2^{n-1}+\mu_n\bar{b}_2^n)\dfrac{\lambda_2}{1-\bar{b}_2z}\\ \vdots\\ \quad+(\mu_0+\mu_1\bar{b}_n+\cdots+\mu_{n-1}\bar{b}_n^{n-1}+\mu_n\bar{b}_n^n)\dfrac{\lambda_n}{1-\bar{b}_nz}.\end{cases}$$