Automatica 50 (2014) 1611–1616

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A computing method on stability intervals of time-delay for fractional-order retarded systems with commensurate time-delays☆

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ARTICLE INFO

Article history:
Received 8 April 2013
Received in revised form
22 February 2014
Accepted 24 February 2014
Available online 6 May 2014

Keywords:
Fractional-order systems
Time-delays
Stability intervals
Crossing frequencies

ABSTRACT

This paper investigates the stability intervals of time-delays for fractional-order retarded time-delay systems. By the Orlando formula, the existence of the crossing frequencies is brought to verify the stability related to the commensurate time-delay. For each crossing frequency, the corresponding critical time-delays are determined by the generalized eigenvalues of two matrices constructed by the crossing frequency, the commensurate fractional-order and the coefficients of the characteristic function. The root tendency (RT) is defined to provide a method to analyze the number of the unstable roots for a given crossing frequency and critical time-delay. Based on the RT values and the number of the unstable roots for fractional-order systems with no time-delay, a computing method on the stability intervals of time-delay is proposed in this paper. Finally, a numerical example is offered to validate the effectiveness of this method.

1. Introduction

Recently, the investigations on fractional-order systems have attracted much attention, due to its more accurate descriptions for real-world systems, especially for systems with the dynamic characteristics of viscoelasticity and diffusion (Krishna, 2011; Machado, Kiryakova, & Mainardi, 2011). Meanwhile, fractional-order controllers have been applied to many engineering applications, by introducing the flexibility in control systems (Efe, 2011). The essential requirement of controller design is to achieve of the stability of any control system. For the fractional-order systems represented by transfer function, the condition that all the characteristic roots of the fractional-order characteristic equation locate at the left half-plane is the stability criterion (Bonnet & Partington, 2002; Matignon, 1998). Based on this criterion, the stability of fractional-order systems with interval uncertainties was discussed by Gao and Liao (2013) and Moornani and Haeri (2010). For state-space description of fractional-order systems with no time-delay, the stability criteria were offered for the commensurate fractional-order α belonging to $0 < \alpha < 1$ and $1 < \alpha < 2$ by Lu and Chen (2010), Farges, Moze, and Sabatier (2010) and Lan and Zhou (2011).

In practical plants including fractional-order systems, the time-delay is a common phenomenon such as in the heating process of the aluminum rod (Victor, Malti, Garnier, & Oustaloup, 2013). Although the requirements of the fractional-order Lyapunov functions were proposed by Li, Chen, and Podlubny (2009) and Baleanu, Ranjbar, Sadati, Delavari, and Abdeljawad (2011), it is not straightforward to establish a specific Lyapunov function, such as the quadratic function for integer-order systems. Thereby, most of the investigations on the stability of time-delay fractional-order systems are based on transfer function models. Bonnet and Partington (2002) and Moornani and Haeri (2011) extended the stability criteria for fractional-order retarded and neutral systems with time-delays in the frequency domain, requiring that all the roots of the characteristic equations lie at the left half-plane. Since the exponential type transcendental term is involved in the characteristic function, an infinite number of characteristic roots exist, leading to the complexity in the stability analysis. To overcome this obstacle, a number of criteria have been presented. Hwang and Cheng (2006) presented a numerical algorithm for the BIBO-stability of fractional-order time-delay systems based on using Cauchy's integral theorem and solving an initial-value problem. Shi and Wang (2011) proposed an analytical criterion for the BIBO-stability of fractional-order time-delay systems by argument formula for the complex function. Yu and Wang (2011) proposed a graphical method of the BIBO-stability on fractional-order systems

☆ This work was supported by the National Natural Science Foundation of China (Grant No. 61304094). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Hitay Ozbay under the direction of Editor Miroslav Krstic.

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http://dx.doi.org/10.1016/j.automatica.2014.03.019

with time-delays was proposed, and stability on the interval uncertainties in the coefficients.

The aforementioned methods are available for the stability analysis with a fix time-delay. To identify all the stability intervals, the numerical methods were proposed for the fractional-order systems with commensurate time-delays by Fioravanti, Bonnet, Ozbay, and Niculescu (2012). In this study, we propose an analytical method to determine the stability of the fractional-order retarded systems with the commensurate time-delays, based on the concept by Fioravanti et al. (2012). By the Orlando formula, the approach to compute the crossing frequencies for integer-order retarded systems with the commensurate time-delays were brought by Chen, Gu, and Nett (1995). We extend this method to find the crossing frequencies for fractional-order time-delay systems. Then, the critical time-delays corresponding to the crossing frequencies are established, and all the stability intervals on the time-delay are determined. Meanwhile, the number of unstable roots can be obtained by an analytical method, avoiding running a great deal of computation.

The rest of the paper is organized as follows. Section 2 addresses the problem formulation and some lemmas. Section 3 brings the computing methods of the crossing frequencies, the critical time-delays and the number of unstable roots. Section 4 offers a numerical example to illustrate the effectiveness of the proposed method. Section 5 concludes the work.

2. Problem formulation

In this study, we compute the stability interval of the time-delay term τ for the following commensurate fractional-order retarded system as:

$$D^{n\alpha}y(t)+\sum_{i=0}^{n-1}\sum_{k=0}^{q}a_{k,i}D^{i\alpha}y(t-k\tau)=u(t), \tag{1}$$

where $y(t)$ and $u(t)$ are the output and control input respectively, the coefficients $a_{k,i}$, $k=0,1,\dots,q$, $i=0,1,\dots,n-1$ are the known real constants, D^{α} is the fractional-order derivative which can be defined by the Riemann–Liouville definition or the Caputo definition, and α is the commensurate fractional-order.

Under the zero initial conditions, the Laplace transforms of $D^{\alpha}f(t)$ are both $s^{\alpha}F(s)$ by the Riemann–Liouville and the Caputo definitions, assuming that the Laplace transform of $f(t)$ is $F(s)$. The characteristic function of the system (1) can be represented as follows:

$$F(s,\mathrm{e}^{-\tau s})=\sum_{k=0}^{q}a_k(s)\mathrm{e}^{-\tau ks}, \tag{2}$$

where

$$a_0(s)=\sum_{i=0}^{n-1}a_{0,i}s^{i\alpha}+s^{n\alpha},$$

$$a_k(s)=\sum_{i=0}^{n-1}a_{k,i}s^{i\alpha},\quad k=1,2,\dots,q.$$

The main object of this paper is to determine the stability interval with respect to τ. The stability criterion is similar to the integer-order systems, i.e. if all the real parts of the solutions of the equation $F(s;\mathrm{e}^{-\tau s})=0$ for $\arg(s)\in(-\pi;\pi]$ are less than zero, the fractional-order system (1) is BIBO-stable. For the sake of convenience, the stability investigated in this paper represents the BIBO-stability.

We substitute $s=\mathrm{j}\omega$ into (2), where j is the imaginary unit. If there exists a frequency $\omega=\omega_c$ fulfilling $F(\mathrm{j}\omega_c,\mathrm{e}^{-\mathrm{j}\omega_c\tau})=0$, the oscillation response of time-delay fractional-order system will be produced. For $\omega_c=0$, the characteristic function becomes $F(0,1)$. The existence of the crossing frequency $\omega_c=0$ requires $\sum_{k=0}^{q}a_{k,0}=0$. In this case, the fractional-order time-delay systems are unstable for all τ, and no stability interval of the time-delay exists, thus it is not necessary to test the stability for this case. We assume $\omega_c\neq 0$, namely, the condition $\sum_{k=0}^{q}a_{k,0}\neq 0$ holds in this paper. Since the coefficients of characteristic function are real numbers, the solutions of characteristic equation are real numbers or complex conjugate numbers, ω_c is the crossing frequency implies that $-\omega_c$ is also a crossing frequency. Hence, we investigate the crossing frequencies for $\omega_c\in\mathbb{R}^+$ in this study. Denoting M as the number of the crossing frequencies, the corresponding crossing frequencies ω_c can be represented by

$$\omega_c\in\{\omega_{c1},\omega_{c2},\dots,\omega_{cM}\}. \tag{3}$$

For a specific ω_{ck}, $k=1,2,\dots,M$, the corresponding time-delay value fulfilling the characteristic equation (2) is defined as the critical time-delay $\tau_{k,j,l}$, where $j=1,2,\dots,N_k$, the constant N_k is a positive number determined by Theorem 2 in the next section, and $l=0,1,\dots,+\infty$ by the periodical property for $\tau_{k,j,l}$.

Based on the definition of $\tau_{k,j,l}$, we obtain $\tau_{k,j,l+1}-\tau_{k,j,l}=2\pi/\omega_{ck}$, which means that we can separate the value $\tau_{k,j,l}$ into $\tau_{k,j,l}=\tau_{k,j,0}+2l\pi/\omega_{ck}$, $l=0,1,\dots,+\infty$. The primary task is to obtain the value $\tau_{k,j,0}$, since other critical time-delays can be deduced by the previous separating operation.

The Orlando formula will be used to compute the crossing frequencies and the corresponding critical time-delays, which is given by

Lemma 1 (*Orlando Formula Young, 1979*)**.** *Let $z_i\in\mathbb{C}$, $i=1,2,\dots,m$ be the characteristic roots of the complex polynomial $\tilde{p}(z)=\sum_{k=0}^{m}p_kz^k$, whose corresponding Schur–Cohn–Fujiwara matrix is defined as K, where $p_k\in\mathbb{C}$ for $k=0,1,\dots,m$ are the coefficients of the polynomial. Then,*

$$\det(K)=|p_m|^{2m}\prod_{i=1}^{m}\prod_{j=1}^{m}(1-z_i\overline{z_j}), \tag{4}$$

where Schur–Cohn–Fujiwara matrix is $K=\overline{p}^{\mathrm{H}}(S)\overline{p}(S)-p^{\mathrm{H}}(S)p(S)$, $p(S)=p_0I+p_1S+\cdots+p_mS^m$, and $S\in\mathbb{R}^{m\times m}$ is the shift matrix defined by

$$S=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\\0&0&0&\dots&0\end{bmatrix},$$

where X^{H} represents the conjugate transpose of the matrix X.

For the sake of the convenience, we define the following matrix C and establish the relationship between the matrices C and K in Lemma 2.

Lemma 2 (*Chen et al., 1995*)**.** *Let $C=\begin{bmatrix}p^{\mathrm{T}}(S)&(\overline{p}^{\mathrm{T}}(S))^{\mathrm{H}}\\\overline{p}^{\mathrm{T}}(S)&(p^{\mathrm{T}}(S))^{\mathrm{H}}\end{bmatrix}$, where*

$$p^{\mathrm{T}}(S)=\begin{bmatrix}p_0&0&\cdots&0\\p_1&p_0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\p_{m-1}&p_{m-2}&\cdots&p_0\end{bmatrix},$$

$$(p^{\mathrm{T}}(S))^{\mathrm{H}}=\begin{bmatrix}p_m&p_{m-1}&\cdots&p_1\\0&p_m&\cdots&p_2\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&p_m\end{bmatrix},$$

we have $\det(K)=(-1)^m\det(C)$, where X^{T} represents the transpose of the matrix X.