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Brief paper

Distributed finite-time consensus of nonlinear systems under switching topologies ☆

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ABSTRACT

In this paper, the finite time consensus problem of distributed nonlinear systems is studied under the general setting of directed and switching topologies. Specifically, a contraction mapping argument is used to investigate performance of networked control systems, two classes of varying topologies are considered, and distributive control designs are presented to guarantee finite time consensus. The proposed control scheme employs a distributed observer to estimate the first left eigenvector of graph Laplacian and, by exploiting this knowledge of network connectivity, it can handle switching topologies. The proposed methodology ensures finite time convergence to consensus under varying topologies of either having a globally reachable node or being jointly strongly connected, and the topological requirements are less restrictive than those in the existing results. Numerical examples are provided to illustrate the effectiveness of the proposed scheme.

1. Introduction

Distributed consensus is a study dedicated to ensuring an agreement between states or output variables among networked systems (Qu, 2009). It is well established that common challenges in this venue are how to achieve consensus with the least possible topological requirement, and how to achieve it in a timely and distributive manner. In this regard, distributed finite time consensus became an instant popular topic in the community, especially with recent advances on finite time stability (Bhat & Bernstein, 2000). Breakthroughs have been made with both continuous (Jiang & Wang, 2009; Khoo, Xie, & Man, 2009; Li, Du, & Lin, 2011; Ou, Du, & Li, 2014; Shang, 2012; Wang & Xiao, 2010; Xiao, Wang, Chen, & Gao, 2009) and discontinuous inputs (Cao & Ren, 2012a,b; Chen, Lewis, & Xie, 2011; Cortés, 2006; Shi & Hong, 2009; Sundaram & Hadjicostis, 2007). To be more precise, finite time consensus with continuous input can be treated as an extension of Bhat and Bernstein (2000) to multi-agent systems, and are in general conducted under time-invariant graph. In particular, it is shown in Jiang and Wang (2009); Wang and Xiao (2010); Xiao et al. (2009) that the graph shall be undirected or directed but detailed-balanced in order to achieve a finite time convergence. This condition is further released in Shang (2012), where finite time consensus is ensured for digraph (i.e., directed graph) with a spanning tree. In addition, applications of continuous finite time consensus have been carried out in formation control of leader–follower multi-agent systems (Li et al., 2011) and nonholonomic robots (Ou et al., 2014), as well as robust finite time tracking of multi-robot systems (Khoo et al., 2009).

Due to the highly nonlinear nature of discontinuous input (i.e, signum/binary protocol), the convergence analysis of networked systems with discontinuous input is extremely challenging, and its solution, if possible, is often sophisticated. For instance, Cortés (2006) pioneered finite time consensus with discontinuous input, under undirected graph, using nonsmooth stability analysis. In Chen et al. (2011), Filippov solution (Filippov, 1960) is introduced for undirected and directed but detailed-balanced networks using a binary protocol and pinning control scheme. The most recent contribution for this topic witnessed the application of a comparison-based Lyapunov approach in both directed (Cao & Ren, 2012b) and undirected (Cao & Ren, 2012a) networks. In addition,

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discrete finite time consensus of time-invariant network is investigated in Sundaram and Hadjicostis (2007) using weighted matrix with minimal polynomial of the smallest degree. Additionally, Shi and Hong (2009) focuses on directed and switching network, and it proves that digraph shall be quasi-strongly connected and contain no direct circle at any interval, in order to ensure a finite time convergence.

However, it should be pointed out that all of the aforementioned results are derived with rather restrictive topological requirements (i.e., undirected graph, digraph but detailed-balanced, or digraph being quasi-strongly connected), finite time consensus of a generic directed network with switching topologies has not received sufficient attention. In this paper, we attempt to solve this problem for a class of nonlinear systems under mild assumptions. The main contribution of this paper is twofold: (i) what are the least conservative topological requirements to ensure a finite time consensus under directed and switching topologies? Is there a simple argument to perform the convergence study of networked systems with discontinuous input? In this paper, we attempt to provide answers to these two questions; and (ii) with the recent advance on network connectivity of a digraph (Qu, Li, & Lewis, 2014), we propose a distributive control scheme that makes finite time consensus possible over any jointly strongly connected network.

2. Preliminaries on graph theory

In this paper, we consider a digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E}$ denote the set of vertices/nodes and the set of directed edges/paths, respectively. Vertex j is said to be adjacent to vertex i if there exists a directed edge $(j, i) \in \mathcal{E}$ with node i being the head and node j being the tail. Analogously, neighborhood set $\mathcal{N}_i \subseteq \mathcal{V}$ of vertex i is $\{k \in \mathcal{V} \mid (k, i) \in \mathcal{E}\}$. Without loss of any generality, adjacency matrix $A(\mathcal{D})$ used in this paper is weighted and normalized as:

$$[A(\mathcal{D})]_{ik} = \begin{cases} a_{ik} > 0 & \text{if } k \neq i,\ (k, i) \in \mathcal{E} \\ 1 - \sum_{k\neq i} a_{ik} & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

That is, matrix $A(\mathcal{D})$ is chosen to be nonnegative and row-stochastic. Furthermore, we assume that the nonzero, and hence positive, weighting factors are all uniformly lower and upper bounded as $\underline{a} \le a_{ij} \le 1$, where $0 < \underline{a} \le 1$, for any $j \in \mathcal{N}_i$. As such, the weighted graph Laplacian is

$$\mathcal{L}(\mathcal{D}) \triangleq I - A(\mathcal{D}) \tag{2}$$

where I is the identity matrix with proper dimension.

Graph $\mathcal{D}$ is said to have *one globally reachable node* if there exists node i such that there is a directed path from node i to node j for all $j \in \mathcal{V}$ with $j \neq i$. Graph $\mathcal{D}$ is called *strongly connected* if there is a directed path between any pair of vertices: a directed path exists from i to k and so does a directed path from k to i for every pair of vertices i, k; or every node is a globally reachable node; or equivalently, Laplacian $\mathcal{L}(\mathcal{D})$ is *irreducible* (Qu, 2009).

According to (2), row sums of Laplacian $\mathcal{L}(\mathcal{D})$ are all zero. It follows that $\lambda_1 = 0$ is the smallest eigenvalue of $\mathcal{L}$ with right eigenvector $\nu_1 \triangleq \frac{1}{\sqrt{n}}\mathbf{1}_n$ and left eigenvector $\gamma = [\gamma_i] \in \Re^n$ defined by

$$\mathcal{L}^T\gamma = 0, \qquad \mathbf{1}_n^T\gamma = 1, \tag{3}$$

where $\mathbf{1}_n \triangleq [1 \ldots 1]^T$, and superscript T denotes matrix transpose. By Perron–Frobenius theorem, all other eigenvalues have positive real parts if the topology of graph $\mathcal{D}$ either has a globally reachable node or is strongly connected. Moreover, as shown in Qu et al. (2014), connectivity (and social standings in the connected network) of $\mathcal{D}$ can be described by left eigenvector γ (and its components). Existing results on γ and its distributed estimation are summarized into the following lemma; its proof is omitted here since it merely combines the results in Qu (2009); Qu et al. (2014).

Lemma 1. *Consider graph Laplacian $\mathcal{L}$ defined by (2) with γ being its left eigenvector defined in (3). Then, the following results hold:*

- *if $\mathcal{D}$ has a global reachable node, γ is unique and non-negative, and $\gamma_i > 0$ (e.g., $\gamma_i = 1$) implies that node i belongs to the leader group[2] (e.g., being a sole leader), and $\gamma_i = 0$ means that node i belongs to the follower group. If $\mathcal{D}$ is strongly connected, $\gamma_i > 0$ for all i;*
- *γ can be estimated distributively at system i by*

$$\dot{\hat{\gamma}}^{(i)}(t) = \sum_{j=1}^{n} a_{ij}(t)\left[\hat{\gamma}^{(j)}(t) - \hat{\gamma}^{(i)}(t)\right] \tag{4}$$

where $\hat{\gamma}^{(i)} \in \Re^n$ is the estimate of γ at system i, $\hat{\gamma}^{(i)}(t_0) = e_i$, $e_i \in \Re^n$ is a vector of zeros except its ith entry being one, and $a_{ij}(t)$ are those defined in (1). Note that $\hat{\gamma}^{(i)}(t) = e_i$ must be reset once any topological switching is detected locally (by examining its corresponding row components of $\mathcal{L}$) and that such resetting should be propagated to the neighbors.

The following lemma will be used in the subsequent technical derivations.

Lemma 2. *Consider Laplacian matrix $\mathcal{L}$ defined in (2) and its left eigenvector γ defined in (3). Then, for any $\mu > 0$ and $t > 0$ and for any $\mathcal{D}$ with a globally reachable node,*

$$e^{\mp\mu\mathcal{L}t} = \mathbf{1}_n\gamma^T + \Gamma_s e^{\mp\mu\Lambda_s t} W_s^T,$$

where Λ_s is the Jordan form associated with eigenvalues λ_2 up to λ_n, $\Gamma_s \in \Re^{n\times(n-1)}$ is the resulting matrix of corresponding right eigenvectors after removing eigenvector $\mathbf{1}_n$ associated with $\lambda_1(\mathcal{L}) = 0$, $W_s \in \Re^{n\times(n-1)}$ consists of all the left eigenvectors of A except for γ.

To consider time-varying topologies, we introduce time sequence $\{t_k : k \in \aleph^+\}$ for $\aleph^+ = \{0, 1, \ldots, \infty\}$, and, without loss of any generality, graph $\mathcal{D}(t)$ is time invariant during interval $t \in [t_k, t_{k+1})$, that is, $A(t_k^+) = A(t_{k+1}^-)$.

3. Finite time consensus under switching topologies with globally reachable node(s)

Consider the network control problem for n nonlinear systems of identical dynamics:

$$\dot{x}_i = f(t, x_i) + u_i, \quad i \in \mathcal{V}, \tag{5}$$

where $x_i \in \Re^m$ is the state of the ith system, $u_i \in \Re^m$ is the neighboring feedback control to be designed, and $f(t, x_i)$ denotes the individual dynamics. For simplicity, $m = 1$ is set in the subsequent technical discussion, and the general case of $m > 1$ can be addressed analogously.

Function $f(t, x_i)$ is assumed to be uniformly bounded with respect to t and locally uniformly bounded with respect to x_i. It is obvious that system (5) is stabilizable, and hence it can be assumed without loss of any generality that, for all $x_i(0) \in \Omega_0$ and with $u_i \equiv 0$, $x_i(t)$ is uniformly bounded as $x_i(t) \in \Omega$ and

$$\|f(t, x_i)\| \le \xi_f, \tag{6}$$

[2] Node i is said to be a leader (or belong to the leader group) if all edges initiated at node i are tails (or for any $j \to i$, node j is also a leader), node i being a follower can be defined analogously.