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Brief paper

Quantitative measure of observability for linear stochastic systems[☆]

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ABSTRACT

In this study we define a new observability measure for stochastic systems: the mutual information between the state sequence and the corresponding measurement sequence for a given time horizon. Although the definition is given for a general system representation, the paper focuses on the linear time invariant Gaussian case. Some basic analytical results are derived for this special case. The measure is extended to the observability of a subspace of the state space, specifically an individual state and/or the modes of the system. A single measurement system represented in the observable canonical form is examined in detail. A recursive form of the observability measure for a finite time horizon is derived. The possibility of using this form for designing a sensor selection algorithm is demonstrated by two examples.

1. Introduction

For deterministic systems, observability is acquired and handled by the rank condition of the observability Gramian matrix (Kalman, 1960). The outcome of the procedure is binary; the system is either completely observable or unobservable. The procedure does not provide any information about the degree of observability. The aim of this study is to analyse the degree of observability of linear Gaussian representations and provide an observability measure for them.

Quantitative measures for the observability of deterministic systems are proposed by Muller and Weber (1972) and Tarokh (1992). Mode observability is examined for deterministic systems by Choi, Lee, and Zhu (1999), Hamdan and Nayfeh (1989), Lindner, Babendreier, and Hamdan (1989) and Porter and Crossley (1970).

Different definitions of the stochastic observability are proposed by Aoki (1967), Bageshwar, Egziabler, Garrard, and Georgiou (2009), Davis and Lasdas (1992), Dragan and Morozan (2006), Han-Fu (1980), Liu and Bitmead (2011), Shen, Sun, and Wu (2013), Ugrinovskii (2003), Van Handel (2009) and Xie, Ugrinovskii, and Petersen (2004). Davis and Lasdas (1992), Van Handel (2009) and Xie et al. (2004) use the probability density functions to define the stochastic observability.

Baram and Kailath (1988) provide an estimability definition that is extended to the definition of the stochastic observability by Liu and Bitmead (2011). Baram and Kailath (1988), Han-Fu (1980) and Liu and Bitmead (2011) use the conditional covariance matrices for the definitions. Ugrinovskii (2003) (for linear stochastic uncertain continuous-time systems) and Liu and Bitmead (2011) (for nonlinear systems) define the observability by using information theoretical approaches.

In addition to the observability definition, some observability measures are also provided in the literature. Kam, Cheng, and Kalata (1987), Mohler and Hwang (1988) and Sujan and Dubowsky (2003) define observability measures by using the information theoretical approaches. Liu and Bitmead (2011) note that the mutual information can be used as the observability measure. Mohler and Hwang (1988) define the observability measure as the mutual information between the state at the last time and the past measurements. Chen, Hu, Li, and Sun (2007) extend the results of Kam et al. (1987) to continuous state systems by using the quantized versions of continuous variables. Hong, Chun, Kwon, and Lee (2008) define the observability measure by using the observability Gramian.

The goal of this paper is to introduce and analyse a new observability measure definition based on the mutual information between the state sequence and the measurement sequence. The definition is given for general nonlinear stochastic systems but is applied to LTI (linear time invariant) discrete time Gaussian

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case to achieve more specific results. Our approach is different from all the studies presented previously since it addresses a state sequence rather than a state at a given time. Although the information gained about the last state is important in some applications, it is also important how well we know the whole state sequence. As an example, we can mention the applications that do batch processing. A possible application area that may require this definition is the dim target tracking that uses batch data. Another very important application area may be the sensor network design, i.e., decisions made on the number, type and placement of the sensors for the optimal state estimation of the moving targets. Two examples are provided to demonstrate such a case. The application areas are not limited to the ones given above. We believe that the new observability definition offered in this study will fill a gap that exists in this area.

An observability measure of "a subspace of the state space" is also defined and analysed in this study. This concept can be expanded to the observability of the modes of the system. In addition, the observability measure based on the state sequence of an individual state is examined in detail for a single measurement system represented in the observable canonical form.

A much shorter version of this paper is presented in 18th IFAC World Congress (Subasi & Demirekler, 2011). Compared with that conference paper, this paper provides more concrete definitions and offers proofs for some of the results. We cannot provide all the proofs due to space limitations but the details can be found in Subasi (2012). Furthermore, the observability measure of a subspace of the state space concept is improved by extending the definition to the partial measurement case. The analysis of the single measurement system represented in the observable canonical form is presented here for the first time. In addition, the observability measure based on the state sequence is expressed recursively and two examples are given to demonstrate the use of the recursive form of the observability measure for sensor selection algorithm design.

Our main contributions are: definition of an observability measure for an interval; analysis of the relationship between the system matrices and the measure; definition and analysis of the subspace (mode) observability measures; analysis of the system given in the observable canonical form that gives some interesting properties of the information flow in the system and presentation of the observability measure in a recursive way which can be used for sensor selection algorithm designs as demonstrated by the examples.

The paper is organized as follows. In Section 2 system under study is defined. Explicit formulae of the observability measure based on the state sequence are obtained in Section 3. The recursive evaluation of the observability measure is also given. In Section 4, the observability measures of a subspace of the state space are given. A single measurement system which is represented in the observable canonical form is examined in detail. In Section 5 two examples are given for sensor selection algorithm design. Section 6 contains the concluding remarks.

2. Preliminaries

The system that we have analysed is represented by the following equations:

$$x_{k+1} = Ax_k + G\omega_k \tag{1}$$

$$y_k = Cx_k + H\upsilon_k \tag{2}$$

where $x_k \in \Re^n$, $y_k \in \Re^m$ and A, G, C, H are constant matrices. It is assumed that $\{x_0,\ w_k \in \Re^p,\ \upsilon_k \in \Re^r\}_{k=0}^{\infty}$ are independent and

$$x_0 \sim N(\bar{x}_0, \Sigma_0) \tag{3}$$

$$\omega_k \sim N(0, Q) \tag{4}$$

$$\upsilon_k \sim N(0, R). \tag{5}$$

We define x_0, $\{\omega_k\}$ and $\{\upsilon_k\}$ as the basic random variables. For time k, the state equation of the system can be written in terms of the basic random variables as:

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} G\omega_i \tag{6}$$

which yields the measurement equation as:

$$y_k = CA^k x_0 + C\sum_{i=0}^{k-1} A^{k-1-i} G\omega_i + H\upsilon_k. \tag{7}$$

Since our goal is to obtain an observability measure for the complete state sequence, the state equations are written in the following form:

$$\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}}_{X^k} = \underbrace{\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^k \end{bmatrix}}_{A_k} x_0 + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ G & 0 & \cdots & 0 \\ AG & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{k-1}G & A^{k-2}G & \cdots & G \end{bmatrix}}_{G_k} \underbrace{\begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{k-1} \end{bmatrix}}_{W^k}. \tag{8}$$

Define

$$H_k = \mathrm{diag}\begin{bmatrix} H & H & H & \cdots & H \end{bmatrix} \tag{9}$$

$$C_k = \mathrm{diag}\begin{bmatrix} C & C & C & \cdots & C \end{bmatrix} \tag{10}$$

$$Q_k = \mathrm{diag}\begin{bmatrix} Q & Q & Q & \cdots & Q \end{bmatrix} \tag{11}$$

$$R_k = \mathrm{diag}\begin{bmatrix} R & R & R & \cdots & R \end{bmatrix} \tag{12}$$

$$Y^k = \begin{bmatrix} y_0 & y_1 & y_2 & \cdots & y_k \end{bmatrix}^T \tag{13}$$

$$V^k = \begin{bmatrix} \upsilon_0 & \upsilon_1 & \upsilon_2 & \cdots & \upsilon_k \end{bmatrix}^T \tag{14}$$

where $(\cdot)^T$ is the transpose operator. Now we can write the state and the measurement equations as:

$$X^k = A_k x_0 + G_k W^k \tag{15}$$

$$Y^k = C_k A_k x_0 + C_k G_k W^k + H_k V^k. \tag{16}$$

The random vectors X^k and Y^k are normal, their mean values are $A_k\bar{x}_0$ and $C_kA_k\bar{x}_0$. Their covariance matrices are given by:

$$\Sigma_{X^k} = A_k \Sigma_0 A_k{}^T + G_k Q_k G_k{}^T \tag{17}$$

$$\Sigma_{Y^k} = C_k \Sigma_{X^k} C_k{}^T + H_k R_k H_k{}^T \tag{18}$$

$$\Sigma_{X^kY^k} = \Sigma_{X^k} C_k{}^T. \tag{19}$$

3. Observability measure for the state sequence

In this section, the definition of the observability measure based on the observability of the complete state sequence is given for LTI discrete-time Gaussian stochastic systems. The mutual information $I(X, Y)$ between the two continuous random variables with a joint density $f(x, y)$ is defined as (Cover & Thomas, 2006):

$$I(X, Y) = \iint f(x, y) \log \frac{f(x, y)}{f(x)f(y)} \mathrm{d}x\mathrm{d}y. \tag{20}$$

Definition 1. The observability measure is the mutual information between the state sequence X^k and the measurement sequence Y^k.