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Solutions and evaluations for fitting of concentric circles

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ABSTRACT

Many practical applications involve the basic problem of fitting a number of data points to a pair of concentric circles, including coordinate metrology, petroleum engineering and image processing. In this paper, two versions of the Levenberg–Marquardt (LM) are applied to obtain the maximum likelihood estimator of the common center and the radii for the concentric circles. In addition, two numerical schemes for conic fitting are extended to the concentric circles fitting problem, as well as several algebraic fits are proposed. This paper shows analytically that the MLE and the numerical schemes are statistically optimal in the sense of reaching the Kanatani–Cramér–Rao (KCR) lower bound, while the other algebraic fits are suboptimal. Our results are confirmed by several numerical experiments.

1. Introduction

A few studies in the literature are available for the fitting of coupled geometric objects, such as concentric circles. Many objects encountered in practice are circularly concentric. A simple example is the inner and outer boundaries of a pipe. Another example is the inner and outer edges of an iris image. The concentric circles fitting methods proposed here have potential use in automatic inspection for pipes, iris recognition for biometric applications, calibration for cameras [14] and ellipticity estimation of steel coils [16].

Dampegama [11] is perhaps the first to introduce the concentric circle estimation problem and propose a solution for obtaining the size of the ruined Abhayagiriya stupa. Benko et al. [4] and O'Leary et al. O' [18] investigated the fitting problem further and developed better solutions. O'Leary et al. O' [18] applied the quadratically constrained total least squares method to solve the fitting problem. In addition to algebraic solution, an iterative method was proposed by Benko et al. [4]. Recently, Ma and Ho [16] developed an explicit solution to coupled circles and ellipses fittings. Their fit is non-iterative and it works well under very general statistical assumption, however their fitting may have heavy bias when data have large level of noise and are sampled along relatively short arcs.

In this paper we develop new solutions for the fitting of concentric circles to a number of data points. The first proposed estimator is the maximum likelihood estimator (MLE). We also extended well-known schemes for conic fitting problem to the problem of fitting concentric circles. These schemes are numerical procedures, such as Renormalization or other schemes that solve the so-called gradient weighted algebraic fit (GRAF). These numerical schemes and the MLE are iterative fits and their performance depend heavily on the initial guess, and supplying a good initial guess is a must.

The best way of choosing the initial guess is the algebraic fit that is obtained by using some approximate of the MLE or GRAF, and as such, two other algebraic fits are proposed in addition to extending one of the most popular fits in single circle/conic fitting that is known as the Taubin fit [20]. However, direct application of the Taubin fit to

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concentric circles is prohibited due to the singularity issue and we have proposed a modification to the Taubin fit to eliminate the singularity problem.

To mathematically formulate our problem, let us first denote n_i to be the number of data points on the ith component of the coupled (concentric) circles, $i=1,2,\dots,K$. The data from the concentric circles are modeled as

$$\mathbf{s}_{ij}=\tilde{\mathbf{s}}_{ij}+\mathbf{n}_{ij},\quad j=1,2,\dots,n_i,\quad i=1,2,\dots,K, \tag{1.1}$$

where $\mathbf{s}_{ij}=(x_{ij},y_{ij})^T$ represents the 2×1 vector containing the Cartesian coordinates of the jth point observed around the ith circle. Also, $\tilde{\mathbf{s}}_{ij}=(\tilde{x}_{ij},\tilde{y}_{ij})^T$ is the true value and $\mathbf{n}_{ij}=(\delta_{ij},\epsilon_{ij})^T$ is the observational noise of $\mathbf{s}_{ij}$. Generally in this paper, we use the common notations that bold capital letters represent matrices and bold lower case letters denote vectors. $\mathbf{1}_m$ and $\mathbf{0}_m$ will be used to denote unity and zero vectors of length m. Also the symbol tilde represents the true value. For an unknown parameter $\bullet$ to be estimated, $\hat{\bullet}$ denotes its estimate while $\bullet$ itself is viewed as a variable for optimization. For ease of illustration we shall consider two concentric circles ($K=2$) in the following. The developed algorithms can be extended to more than two concentric circles in a direct manner and we provide some details of the extension at the end of Section 4.

The true point $\tilde{\mathbf{s}}_{ij}$ satisfies the following relation:

$$P_i(\tilde{\boldsymbol{\theta}},\tilde{\mathbf{s}}_{ij}):=\|\tilde{\mathbf{s}}_{ij}-\tilde{\mathbf{c}}\|^2-\tilde{R}_i^2=0. \tag{1.2}$$

The 4-dimensional vector $\tilde{\boldsymbol{\theta}}$ is the vector of the true parameters $\tilde{\boldsymbol{\theta}}=(\tilde{\mathbf{c}}^T,\tilde{R}_1,\tilde{R}_2)^T$, where $\tilde{\mathbf{c}}=(\eta,\tilde{b})^T$ is the common circle center, $\tilde{R}_1$ and $\tilde{R}_2$, with $\tilde{R}_1<\tilde{R}_2$, are the two circle radii, and $\|*\|$ is the Euclidean norm. In this paper, $\tilde{\boldsymbol{\theta}}$ is the unknown parameter vector to be estimated.

Expanding the square in Eq. (1.2) and denoting $\|\tilde{\mathbf{s}}_{ij}\|^2$ as $\tilde{z}_{ij}$, Eq. (1.2) can be expressed as

$$P_i(\tilde{\boldsymbol{\phi}},\tilde{\mathbf{s}}_{ij})=\tilde{A}\tilde{z}_{ij}+\tilde{B}\tilde{x}_{ij}+\tilde{C}\tilde{y}_{ij}+\tilde{D}_i=0, \tag{1.3}$$

where $\tilde{\boldsymbol{\phi}}=(\tilde{A},\tilde{B},\tilde{C},\tilde{D}_1,\tilde{D}_2)^T$ is the algebraic parameter vector. Thus, one might alternatively estimate the algebraic parameter vector, then recover the natural parameters through the relationships between the two parametric spaces

$$a=\frac{-B}{2A},\quad b=\frac{-C}{2A},\quad R_i^2=\frac{B^2+C^2-4AD_i}{4A^2},\ i=1,2. \tag{1.4}$$

Eq. (1.3) can be expressed in a compact form as $P_i=\tilde{\boldsymbol{\phi}}^T\tilde{\mathbf{z}}_{ij}$, where

$$\tilde{\mathbf{z}}_{1j}=\left(\tilde{z}_{1j},\tilde{x}_{1j},\tilde{y}_{1j},1,0\right)^T \text{ and } \tilde{\mathbf{z}}_{2j}=\left(\tilde{z}_{2j},\tilde{x}_{2j},\tilde{y}_{2j},0,1\right)^T. \tag{1.5}$$

For notation simplicity, we shall collect all $n=n_1+n_2$ measurement data points together and represent them as

$$\mathbf{s}=\tilde{\mathbf{s}}+\mathbf{n}, \tag{1.6}$$

where $\mathbf{s}=(\mathbf{s}_{11}^T,\dots,\mathbf{s}_{1n_1}^T,\mathbf{s}_{21}^T,\dots,\mathbf{s}_{2n_2}^T)^T$ and

$$\mathbf{n}=\left(\mathbf{n}_{11}^T,\dots,\mathbf{n}_{1n_1}^T,\mathbf{n}_{21}^T,\dots,\mathbf{n}_{2n_2}^T\right)^T$$

are $2n\times 1$ vectors of the observed points and the noise vector, respectively; while $\tilde{\mathbf{s}}$ is the true value of $\mathbf{s}$.

We also shall model $\mathbf{n}$ as zero-mean Gaussian with a covariance matrix equal to $\mathbf{Q}$. In most cases, it is reasonable to assume $\mathbf{n}_{ik}$ and $\mathbf{n}_{jl}$, for $k=l$ or $i\neq j$, are independent so that $\mathbf{Q}$ is a diagonal matrix. Here we consider the specific case that $\text{cov}(\mathbf{n}_{ij},\mathbf{n}_{il})=\sigma^2\hat{\delta}_{jl}$ for all $l,j=1,\dots,n_i$ and $i=1,2$, where $\hat{\delta}_{jl}$ represents the Kronecker Delta function.

The paper is organized as follows. Section 2 discusses the MLE in full details. In Section 3, the gradient weighted algebraic fit (GRAF) and its implementations are discussed. Section 4 focuses on the non-iterative algebraic fits. Section 5 provides the first order error analysis, which is supported with numerical experiments in Section 6.

2. Maximum likelihood estimator

Based on our statistical assumptions, the MLE, $\tilde{\boldsymbol{\theta}}_m$, turns out to be the minimizer of

$$\mathcal{F}_1(\boldsymbol{\theta})=\sum_{i=1}^{2}\sum_{j=1}^{n_i}\|\mathbf{s}_{ij}-\tilde{\mathbf{s}}_{ij}\|^2, \tag{2.1}$$

subject to a system of equations given in Eq. (1.2). This is equivalent to minimizing the sum of the squares of the orthogonal distances of the observed points $\mathbf{s}_{ij}$ to the fitted circles indexed by (a,b,R_1) and (a,b,R_2).

$$\mathcal{F}(\boldsymbol{\theta})=\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(\|\mathbf{r}_{ij}\|-R_i\right)^2=\sum_{i=1}^{2}\sum_{j=1}^{n_i}d_{ij}^2, \tag{2.2}$$

where $\mathbf{r}_{ij}=\mathbf{s}_{ij}-\mathbf{c}$. The signed distance d_{ij} stands for the distance from $\mathbf{s}_{ij}$ to the circle P_i. i.e., for each $i=1,2$, and $j=1,\dots,n_i$,

$$d_{ij}=r_{ij}-R_i,\quad r_{ij}=\|\mathbf{r}_{ij}\|=\sqrt{(x_{ij}-a)^2+\left(y_{ij}-b\right)^2}. \tag{2.3}$$

This result follows from implementing the Lagrangian multipliers λ_{ij}. That is, let

$$\mathcal{F}_2(\boldsymbol{\theta})=\sum_{i=1}^{2}\sum_{j=1}^{n_i}\|\mathbf{s}_{ij}-\tilde{\mathbf{s}}_{ij}\|^2-\sum_{i=1}^{2}\sum_{j=1}^{n_i}\lambda_{ij}\left(\|\tilde{\mathbf{s}}_{ij}-\mathbf{c}\|^2-R_i^2\right). \tag{2.4}$$

Differentiating with respect to $\mathbf{c},R_1,R_2$, and $\tilde{\mathbf{s}}_{ij}$ and then equating the results to zero give

$$\frac{\partial\mathcal{F}_2}{\partial\mathbf{c}}=2\sum_{i=1}^{2}\sum_{j=1}^{n_i}\lambda_{ij}(\tilde{\mathbf{s}}_{ij}-\mathbf{c})=\mathbf{0} \tag{2.5}$$

$$\frac{\partial\mathcal{F}_2}{\partial R_i}=2\sum_{j=1}^{n_i}\lambda_{ij}R_i=0,\quad i=1,2 \tag{2.6}$$

$$\frac{\partial\mathcal{F}_2}{\partial\tilde{\mathbf{s}}_{ij}}=-2\left((\mathbf{s}_{ij}-\tilde{\mathbf{s}}_{ij})+\lambda_{ij}(\tilde{\mathbf{s}}_{ij}-\mathbf{c})\right)=\mathbf{0}\,,. \tag{2.7}$$

From Eq. (2.7), we obtain

$$\tilde{\mathbf{s}}_{ij}=\frac{\mathbf{s}_{ij}-\lambda_{ij}\mathbf{c}}{1-\lambda_{ij}}. \tag{2.8}$$

However, $\tilde{\mathbf{s}}_{ij}$ satisfies (1.2). Substituting (2.8) in (1.2) gives

$$R_i^2=\left\|\frac{\mathbf{s}_{ij}-\lambda_{ij}\mathbf{c}}{1-\lambda_{ij}}-\mathbf{c}\right\|^2=\left\|\frac{\mathbf{r}_{ij}}{1-\lambda_{ij}}\right\|^2. \tag{2.9}$$

This means that $|1-\lambda_{ij}|=R_i^{-1}\|\mathbf{r}_{ij}\|$, and as such, $\lambda_{ij}=1\pm R_i^{-1}\|\mathbf{r}_{ij}\|$. But $\lambda_{ij}=1+R_i^{-1}\|\mathbf{r}_{ij}\|$ cannot be the solution,