Automatica 49 (2013) 1442–1448

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Constructive immersion and invariance stabilization for a class of underactuated mechanical systems ☆

I. Sarras [a], J.Á. Acosta [b,1], R. Ortega [c], A.D. Mahindrakar [d]

[a] Control Engineering and Systems Analysis Department (SAAS), Université Libre de Bruxelles, Brussels 1050, Belgium
[b] Depto. de Ingeniería de Sistemas y Automática, Escuela Técnica Superior de Ingeniería, 41092 Sevilla, Spain
[c] Laboratoire des Signaux et Systèmes, Supelec, Plateau du Moulon, 91192 Gif-sur-Yvette, France
[d] Department of Electrical Engineering, Indian Institute of Technology Madras, Chennai-600036, India

ARTICLE INFO

Article history:
Received 14 June 2011
Received in revised form
30 September 2012
Accepted 27 December 2012
Available online 5 March 2013

Keywords:
Nonlinear control systems
Stabilization
Invariant manifolds
System immersion
Mechanical systems

ABSTRACT

A constructive approach to stabilize a desired equilibrium for a class of underactuated mechanical systems, which *obviates the solution* of partial differential equations, is proposed. The Immersion & Invariance methodology is adopted, with the main result formulated in the Port-Hamiltonian framework, for both model and target dynamics. The procedure is applicable to mechanical systems with under-actuation degree larger than one, extending the results recently reported by some of the authors. The approach is successfully applied to two benchmark examples and some basic connections with the interconnection and damping assignment passivity-based control are revealed. An additional contribution of this work is the identification of a class of mechanical systems whose mechanical structure remains invariant under partial feedback linearization.

1. Introduction

Stabilization of mechanical systems has been a central problem in the control community for several decades. Many research groups have been active in studying such systems, proposing different control laws and applying them to various examples. Among the systematic control design techniques it is worth mentioning those based on Controlled Lagrangians (CL) (Bloch, Leonard, & Marsden, 2000) and the Interconnection and Damping Passivity-Based Control (IDA-PBC) (Ortega, Spong, Gómez-Estern, & Blankenstein, 2002) — see Blankenstein, Ortega, and van der Schaft (2002) for their connection. The applicability for both methods is restricted by the need to solve a set of partial differential equations (PDEs). In Acosta, Ortega, Astolfi, and Mahindrakar (2005) explicit solutions of the PDEs of IDA-PBC are given for a class of mechanical systems with under-actuation degree (or co-dimension) one, *i.e.*, the number of available actuators is less than the number of degrees of freedom by one. To the best of our knowledge, no similar *constructive* result is available for systems with higher under-actuation degree.

With the motivation to avoid the need of solving PDEs, in Acosta, Ortega, Astolfi, and Sarras (2008) the use of the Immersion and Invariance (I&I) methodology for the problem of stabilization of mechanical systems was explored, and a successful application to the cart–pendulum system was reported. The I&I approach for the stabilization of nonlinear systems was introduced in Astolfi and Ortega (2003) and developed further in a series of publications that have been recently summarized in Astolfi, Karagiannis, and Ortega (2007). In the I&I approach the desired behaviour of the system to be controlled is captured by the choice of a target dynamical system of lower dimension than the original system. The control objective is to find a controller, which guarantees that the closed-loop system asymptotically behaves like the target system—achieving in this way *asymptotic* model matching. This should be contrasted with the more restrictive *exact* matching techniques of the CL and IDA-PBC methodologies. This is formalized by finding a manifold in the extended state-space that can be rendered invariant and attractive, with internal dynamics a copy of the desired closed-loop dynamics, and designing a control law that steers the state of the

☆ The work of J.Á. Acosta was supported by the Spanish Ministerio de Educación (MEDU) under grant PR2010-0036 and by the Consejería de Innovación Ciencia y Empresa under the IAC programme (Spain). The material in this paper was partially presented at the 8th IFAC conference on Nonlinear Control Systems NOLCOS 2010, September 1–3, 2010, Bologna, Italy. This paper was recommended for publication in revised form by Associate Editor Warren E. Dixon under the direction of Editor Andrew R. Teel. I. Sarras acknowledges the financial support of the IAP Programme initiated by the Belgian State, Science Policy Office, through the Belgian Network DYSCO. The authors thank graduate student Sai Pushpak for implementing the control law on the IWP experimental setup at the Dynamics and Control Laboratory, IIT Madras. The scientific responsibility rests with the authors.

E-mail addresses: sarras@ieee.org (I. Sarras), jaar@us.es (J.Á. Acosta), ortega@lss.supelec.fr (R. Ortega), arun_dm@iitm.ac.in (A.D. Mahindrakar).

[1] Tel.: +34 954482291; fax: +34 954487340.

0005-1098/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2013.01.059

system towards the manifold. The main idea introduced in Acosta et al. (2008) is to leave some of the parameters of the target dynamics *free*, namely the potential energy function and the damping injection matrix, and use them as degrees of freedom to solve the PDEs. In this work we pursue this line of research, with particular emphasis in providing constructive solutions for mechanical systems with an even number of configuration variables and under-actuation degree larger than one.

The paper is organized as follows. Section 2 presents a version of I&I for the case where the original and the target systems are mechanical systems described in Port-Hamiltonian (pH) form. The design procedure is described in Sections 3 and 4. In Section 5 the results are illustrated with two physical examples: the inertia wheel pendulum and the inverted pendulum on a cart, where experimental results of the former are reported. The paper is wrapped-up with a section on conclusions. Appendices A and B are devoted to identify a class of mechanical systems for which partial linearization transforms the system to the form required by the paper.

2. Port-Hamiltonian formulation of I&I for underactuated mechanical systems

In this section we present the main stabilization result of Astolfi and Ortega (2003) when both, the model and the target dynamics, are mechanical systems given in pH form. Thus, we consider n degrees-of-freedom under-actuated mechanical systems modelled as

$$\Sigma : \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \mathbb{J} \begin{bmatrix} \nabla_q H^\top \\ \nabla_p H^\top \end{bmatrix} + \mathbb{G}(q)u, \quad \mathbb{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{1}$$

where $\mathbb{G}(q) := [0 \; G(q)]^\top$, $q, p \in \mathbb{R}^n$ are the generalized positions and momenta, respectively, $u \in \mathbb{R}^m$ are the control inputs, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the total energy of the system, given as

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q), \tag{2}$$

where $M = M^\top > 0$ and V is the potential energy function, $G \in \mathbb{R}^{n\times m}$ has constant rank $m < n$. Following the main result of Astolfi and Ortega (2003), we state the next theorem for the class of pH mechanical systems that admit as target system another (lower-order) pH mechanical system.

Proposition 1. *Consider the system (1), with $x = \mathrm{col}(q,p)$ and an equilibrium point $x^\star = \mathrm{col}(q^\star, 0) \in \mathbb{R}^{2n}$ to be stabilized. Let $s < n$ and assume we can find mappings*

$$M_t : \mathbb{R}^s \to \mathbb{R}^{s\times s}; \qquad V_t : \mathbb{R}^s \to \mathbb{R}; \qquad \mathbb{R}^{2s} \to \mathbb{R}^{s\times s};$$
$$\pi : \mathbb{R}^{2s} \to \mathbb{R}^{2n}; \qquad \mathbb{R}^{2s} \to \mathbb{R}^m;$$
$$\mu : \mathbb{R}^{2n} \to \mathbb{R}^{2(n-s)}; \qquad \psi : \mathbb{R}^{2n} \times \mathbb{R}^{2(n-s)} \to \mathbb{R}^m;$$

such that the following hold.

(H1) *(Target system) The system*

$$\Sigma_t : \begin{bmatrix} \dot{\xi}_q \\ \dot{\xi}_p \end{bmatrix} = (\mathbb{J}_t - \mathbb{R}_t) \begin{bmatrix} \nabla_{\xi_q} H_t^\top \\ \nabla_{\xi_p} H_t^\top \end{bmatrix},$$
$$\mathbb{J}_t := \begin{bmatrix} 0 & I_s \\ -I_s & 0 \end{bmatrix}, \tag{3}$$

where $\mathbb{R}_t(\xi) := \mathrm{diag}(0, R_t(\xi))$, state $\xi := \mathrm{col}(\xi_q, \xi_p)$, $\xi_q, \xi_p \in \mathbb{R}^s$, the Hamiltonian $H_t : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$

$$H_t(\xi_q, \xi_p) = \frac{1}{2} \xi_p^\top M_t^{-1}(\xi_q) \xi_p + V_t(\xi_q), \tag{4}$$

has an asymptotically stable equilibrium at $\xi^\star = \mathrm{col}(\xi_q^\star, 0) \in \mathbb{R}^{2s}$ and

$$x^\star = \pi(\xi^\star). \tag{5}$$

(H2) *(Immersion condition)*[2] *For all $\xi \in \mathbb{R}^{2s}$*

$$\left[\mathbb{J}\nabla H^\top(x) + \mathbb{G}(x)c(x)\right]|_{x=\pi(\xi)} = \nabla\pi(\xi)(\mathbb{J}_t - \mathbb{R}_t(\xi))\nabla H_t^\top. \tag{6}$$

(H3) *(Implicit manifold) The following set identity holds*

$$\{x \in \mathbb{R}^{2n} | \mu(x) = 0\} = \{x \in \mathbb{R}^{2n} | x = \pi(\xi) \text{ for some } \xi \in \mathbb{R}^{2s}\}. \tag{7}$$

(H4) *(Manifold attractivity and trajectory boundedness) All trajectories of the system*

$$\dot{z} = \nabla\mu(x)\left[\mathbb{J}\nabla H^\top + \mathbb{G}\psi(x,z)\right], \tag{8}$$
$$\dot{x} = \mathbb{J}\nabla H^\top + \mathbb{G}\psi(x,z), \tag{9}$$

are bounded and satisfy $\lim_{t\to\infty} z(t) = 0$.

Then, $x_\star$ is an asymptotically stable equilibrium of the system (1) in closed-loop with the control $u = \psi(x, \mu(x))$.

In the next section the degrees of freedom provided by the target dynamics are used to solve the PDEs (6) from (H2) and in Section 4 we take care of the remaining conditions of Proposition 1.

3. Explicit solution of the PDEs of the immersion condition

The assumption below identifies the class of systems that we consider in the paper. To streamline the assumptions, we introduce the partition $q = \mathrm{col}(q_1, q_2)$, $q_i \in \mathbb{R}^{\frac{n}{2}}$, $i = 1, 2$.

Assumption A.1. The system (1) satisfies the following conditions

(C1) $n = 2m$.

(C2) The inertia matrix is block diagonal, i.e. $M := \mathrm{diag}(M_{11}, M_{22})$ with $M_{ii} \in \mathbb{R}^{m\times m}$, $i = 1, 2$ constant.

(C3) The potential energy function and the input matrix depend only on one of the coordinates q_i, $i = 1, 2$. Without loss of generality (see below), we assume $i = 1$, i.e., $V = V(q_1)$ and $G = G(q_1)$.

The following remarks are in order.

- Condition (C1) is introduced to obtain square matrices that simplify the notation in the sequel—avoiding the need of pseudo-inverses.
- The assumption of constant, block-diagonal, inertia matrix is clearly restrictive. However, there are cases where either a change of coordinates (Venkataraman, Ortega, Sarras, & van der Schaft, 2010) or a partial feedback linearization (Spong, 1998) transforms the system into this form. Moreover, in Appendix A we identify a class of systems for which partial feedback linearization yields another mechanical system that—as is well-known—is in general not the case.
- Condition (C3) is a technicality, needed to obtain explicit expressions—notice that a similar assumption is made in Acosta et al. (2005). As shown, the choice $i = 1$ in (C3) is done without loss of generality. The case where the functions depend on q_2 can be handled with a suitable redefinition of the mapping π in (10).

[2] When clear from the context the argument of the differential operator ∇ is omitted.