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Brief paper

A general maximum principle for optimal control of forward–backward stochastic systems☆

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ABSTRACT

A general maximum principle for optimal control problems derived by forward–backward stochastic systems is established, where control domains are non-convex and forward diffusion coefficients explicitly depend on control variables. These optimal control problems have broad applications in mathematical finance and economics such as the recursive mean–variance portfolio choice problems. The maximum principle is applied to study a forward–backward linear-quadratic optimal control problem with a non-convex control domain; an optimal solution is obtained.

1. Introduction

Since the introduction of nonlinear backward stochastic differential equations (BSDEs) (Pardoux & Peng, 1990), there has been an increasing research interest about optimal control derived by BSDEs or forward–backward stochastic differential equations (FBSDEs). See e.g. the references Bahlali, Boulekhrass, and Mezerdi (2011), Dokuchaev and Zhou (1999), El Karoui, Peng, and Quenez (1997), Huang, Li, and Wang (2010); Huang, Wang, and Xiong (2009), Meng (2009), Peng (1993), Shi and Wu (2010), Wang and Wu (2009), Wang and Yu (2010, 2012), Wu (2010) and Xu (1995). Although there exist lots of works about maximum principle for FBSDEs control system, yet the related general case, i.e., the control domain is not necessarily convex and the forward diffusion coefficient explicitly depends on some control variables, is still an open and unsolved problem (Peng, 1999). From the author's viewpoint, the main difficulty in solving this open problem is how to use a suitable variational technique to treat a backward control system with an extra variable z. Because z is different from the other state variables x and y, we cannot deal with it using some standard variational techniques. Then a certain new idea is expected to be found.

Recently, Kohlmann and Zhou (2000) studied the relationship between a BSDE and a forward linear-quadratic (LQ) optimal control problem. They regard the martingale term in the BSDE as a control and then introduce a new stochastic control problem with the same dynamics as the BSDE but a definite given initial state, in which the optimization goal is to minimize the second moment of the difference between the terminal state and the terminal value given in the BSDE. Based on Kohlmann and Zhou (2000), Lim and Zhou (2001) investigated a backward LQ optimal control problem.

Inspired by Kohlmann and Zhou (2000) and Lim and Zhou (2001), this paper focuses on studying the open problem in Peng (1999). The rest of this paper is organized as follows. In Section 2, the problem is formulated. Section 3 establishes a general maximum principle for the optimal control problem. Section 4 studies an LQ case with a non-convex control domain. The general maximum principle established in this paper is applied to find an optimal solution. Section 5 lists some concluding remarks.

2. Formulation of the optimal control problem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a filtered complete probability space, on which an $\mathbb{R}$-valued standard Brownian motion $(B_t)_{t\ge0}$ is defined with $\mathcal{F}_t$ being its natural filtration, $\mathcal{F} = \mathcal{F}_T$, and T is a constant. The case of multi-dimensional standard Brownian motion is similar, and hence is omitted for simplicity.

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2.1. Motivation

Suppose there are two kinds of securities in a market for possible investment choices. (i) A risk-free security (e.g. a bond): $dS_0(t) = r(t)S_0(t)dt$, $S_0(0) > 0$, where $r(t)$ is a bounded deterministic function. (ii) A risky security (e.g. a stock): $dS_1(t) = \mu(t)S_1(t)dt + \sigma(t)S_1(t)dB_t$, $S_1(0) > 0$, where $\mu(t), \sigma(t) \neq 0$ are bounded deterministic functions, and $\mu(t) > r(t)$ is required. Let $v(t)$ denote the amount invested in the risky security. Given the initial wealth $x(0) = x_0$, the wealth dynamics is

$$\begin{cases} dx(t) = [r(t)x(t) + (\mu(t) - r(t))v(t)]dt + \sigma(t)v(t)dB_t, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $v(t)$ can be negative which means short-selling of the stock. Usually, it is reasonable that there are some constraints on the range of values of $v(t)$. For example, let $v(t)$ belong to $\mathcal{U} = (-\infty, -1] \cup [1, +\infty)$. It implies that there is a minimum constraint for $v(t)$ in the market. Let $\mathcal{U}_{\mathrm{ad}}$ be the set of admissible portfolios valued in $\mathcal{U} \subset \mathbb{R}$, $a > 0$, $b > 0$ and Q are constants, $\beta(\cdot)$ and $\gamma(\cdot)$ are deterministic functions. Then the mean–variance portfolio choice problem is to find an admissible $v^*(\cdot)$ such that $J(v^*(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{\mathrm{ad}}} \frac{1}{2}\mathbb{E}[(x(T)-a)^2+(y(0)-b)^2]$ subject to (2.1)–(2.2), where y is a recursive utility from wealth x, which is the solution of

$$\begin{cases} -dy(t) = (-\beta(t)y(t) + \gamma(t)v(t))dt - z(t)dB_t, \\ y(T) = Qx(T). \end{cases} \tag{2.2}$$

See e.g. El Karoui et al. (1997) for more information. Clearly, this is an LQ optimal control problem of an FBSDE system with a controlled diffusion and non-convex control domain.

2.2. Problem A

Consider the controlled FBSDE

$$\begin{cases} dx(t) = g(t, x(t), v(t))dt + \sigma(t, x(t), v(t))dB_t, \\ dy(t) = f(t, x(t), y(t), z(t), v(t))dt + z(t)dB_t, \\ x(0) = a, \qquad y(T) = \Phi(x(T)), \end{cases} \tag{2.3}$$

where $(x(\cdot), y(\cdot), z(\cdot)) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, $v(\cdot)$ is the control process, and $g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^k \longrightarrow \mathbb{R}^m$, $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Let $\mathcal{U}$ be a non-empty subset of $\mathbb{R}^k$, which is not necessarily convex. A control variable $v(\cdot)$ is called admissible, if $v(t)$ is an $\mathcal{F}_t$-adapted process valued in $\mathcal{U}$ and satisfies $\sup_{0\le t\le T} \mathbb{E}|v(t)|^\iota < +\infty$, $\forall \iota = 1, 2, \ldots$. The set of all admissible controls is denoted by $\mathcal{U}_{\mathrm{ad}}$. For any $v(\cdot) \in \mathcal{U}_{\mathrm{ad}}$, the cost functional is in the form of

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T l(t, x(t), y(t), z(t), v(t))dt + h(x(T)) + \gamma(y(0))\right], \tag{2.4}$$

where $l : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^k \longrightarrow \mathbb{R}$, $h : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\gamma : \mathbb{R}^m \longrightarrow \mathbb{R}$. The optimal control problem under consideration in this paper is

Problem A. To find an admissible $v^*(\cdot)$ such that

$$J(v^*(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{\mathrm{ad}}} J(v(\cdot)). \tag{2.5}$$

If such a $v^*(\cdot)$ attains the infimum, then we call it an optimal control. Eq. (2.3) is called the optimal state equation, and the solution $(x^*(\cdot), y^*(\cdot), z^*(\cdot))$ corresponding to $v^*(\cdot)$ is called an optimal trajectory.

The objective of this paper is to establish a general maximum principle of Problem A. Note that Problem A is related to Bahlali (2008), where g, σ, f, l are bounded and $\mathcal{U}$ is compact. It is well known that the adjoint equation plays an important role in deriving maximum principle. Since $\mathcal{U}$ is not convex and v enters into σ, it is necessary to formulate a suitable second-order adjoint equation for the backward system. However, the backward one is so complicated that it is difficult to construct such an equation. Thus the classical method cannot be directly used here. As found in Kohlmann and Zhou (2000) and Lim and Zhou (2001), z can be regarded as a control variable while $y(T) = \Phi(x(T))$ in (2.3) a terminal state constraint. Fix v, we can choose the control z as well as $y(0)$ so that $y(T)$ exactly hits the target $\Phi(x(T))$. Inspired by this, we formulate a variation of Problem A below.

Problem B. Minimize

$$J(y_0, u(\cdot), v(\cdot)) = \mathbb{E}\left[\int_0^T l(t, x(t), y(t), u(t), v(t))dt + h(x(T)) + \gamma(y_0)\right] \tag{2.6}$$

over $y_0 \in \mathbb{R}^m$, $u(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, $v(\cdot) \in \mathcal{U}_{\mathrm{ad}}$ subject to the *forward* control system

$$\begin{cases} dx(t) = g(t, x(t), v(t))dt + \sigma(t, x(t), v(t))dB_t, \\ dy(t) = f(t, x(t), y(t), u(t), v(t))dt + u(t)dB_t, \\ x(0) = a, \qquad y(0) = y_0, \end{cases} \tag{2.7}$$

with an *optimal* state constraint

$$\mathbb{E}|y(T) - \Phi(x(T))|^2 = 0. \tag{2.8}$$

Note that Problem A is embedded into Problem B, according to the optimal control $(y_0^*, u^*(\cdot), v^*(\cdot))$ of Problem B, we know that $v^*(\cdot)$ is an optimal control of Problem A, y_0^* is the initial value of the optimal trajectory $y^*(\cdot)$, and $u^*(\cdot) = z^*(\cdot)$ in (2.3). However, Problem B can be solved by using the classical second-order variational technique (Peng, 1990).

3. A general maximum principle

Let us impose some assumption conditions on the coefficients of Problem A.

Hypothesis (H). (i) f, g, σ, Φ, l, h and γ are twice continuously differentiable with respect to (x, y, z). (ii) The derivatives up to order 2 of f, g, σ and Φ with respect to (x, y, z) are bounded. (iii) $f, g, \sigma, \Phi, l_x, l_y, l_z, h_x$ and γ_y grow linearly about (x, y, z, v) and is continuous in (t, v). (iv) $l_{xx}, l_{xy}, l_{xz}, l_{yz}, l_{yy}, l_{zz}, h_{xx}$ and γ_{yy} are bounded.

We now try to solve Problem B. Define a metric $d(\cdot,\cdot)$ in $\mathcal{U}_{\mathrm{ad}}$ and $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, $d(u(\cdot), v(\cdot)) \doteq \mathbb{E}[\mathrm{mes}\{t \in [0,T], u(t) \neq v(t)\}]$, $\forall u(\cdot), v(\cdot) \in \mathcal{U}_{\mathrm{ad}}$ or $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, where *mes* denotes the Lebesgue measure. Then it is easy to verify that $(\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m), d(\cdot,\cdot))$ or $(\mathcal{U}_{\mathrm{ad}}, d(\cdot,\cdot))$ is a complete metric space.

Suppose $(y_0^*, u^*(\cdot), v^*(\cdot))$ is an optimal control of Problem B and $(x^*(\cdot), y^*(\cdot))$ is the corresponding optimal state trajectory of (2.7) which satisfies $y^*(T) = \Phi(x^*(T))$. Then it is easy to see that $J(y_0^*, u^*(\cdot), v^*(\cdot)) \le J(y_0, u(\cdot), v(\cdot))$ for any $y_0 \in \mathbb{R}^m$, $u(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, $v(\cdot) \in \mathcal{U}_{\mathrm{ad}}$. For any $(y_0, u(\cdot), v(\cdot)) \in \mathbb{R}^m \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \times \mathcal{U}_{\mathrm{ad}}$, we let $(x^v(\cdot), y^v(\cdot))$ be the solution of (2.7). Note that $(x^v(\cdot), y^v(\cdot))$ does not necessarily satisfy $y^v(T) = \Phi(x^v(T))$ here. Define a new cost functional J_ρ by

$$J_\rho(y_0, u(\cdot), v(\cdot)) \doteq \{[J(y_0, u(\cdot), v(\cdot)) - J(y_0^*, u^*(\cdot), v^*(\cdot)) + \rho]^2 + [\mathbb{E}|y^v(T) - \Phi(x^v(T))|^2]^2\}^{\frac{1}{2}}, \tag{3.1}$$