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On the periodic trajectories of Boolean control networks☆

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ABSTRACT

In this note we first characterize the periodic trajectories (or, equivalently, the limit cycles) of a Boolean network, and their global attractiveness. We then investigate under which conditions all the trajectories of a Boolean control network may be forced to converge to the same periodic trajectory. If every trajectory can be driven to such a periodic trajectory, this is possible by means of a feedback control law.

1. Introduction

Boolean networks (BNs) are state-space models whose state variables attain two possible values (0 and 1, true or false) and whose update is governed by logic functions. The recent interest in BNs is motivated by the large number of natural and artificial systems whose describing variables display only two distinct configurations. Originally introduced to model simple neural networks, BNs recently proved to be suitable to describe and simulate the behavior of genetic regulatory networks (Kauffman, 1969; Shmulevich, Dougherty, Kim, & Zhang, 2002). In addition, BNs are fruitfully used to describe the interactions among agents and hence to investigate consensus problems (Green, Leishman, & Sadedin, 2007; Lou & Hong, 2010). Boolean control networks (BCNs) were subsequently introduced in the literature to keep into account that many biological systems have exogenous inputs. So, by adding Boolean inputs to a BN, it is possible to formally define a BCN. Indeed, a BCN can be seen as a switched system, switching among different BNs.

In addition to the increasingly large number of applications where BNs and BCNs proved their effectiveness, another reason for their recent success is the powerful algebraic framework, developed by D. Cheng and co-authors (Cheng, 2009; Cheng & Qi, 2010; Cheng, Qi, & Li, 2011), where both BNs and BCNs can be recast. The main idea underlying this approach is that a Boolean network with n state variables exhibits 2^n possible configurations, and if any such configuration is represented by means of a canonical vector of size 2^n, all the logic maps that regulate the state-updating can be equivalently described by means of $2^n \times 2^n$ Boolean matrices. As a result, every Boolean network can be described as a discrete-time linear system. In a similar fashion, a Boolean control network can be converted into a discrete-time bilinear system or, more conveniently, it can be seen as a family of BNs, each of them associated with a specific value of the input variables.

In this paper, we investigate the periodic structure of the state trajectories of BNs and BCNs. In detail, we first characterize the periodic trajectories (or, equivalently, the limit cycles) of a Boolean network, and their global attractiveness. We then investigate under which conditions all the trajectories of a BCN may be forced to converge to the same periodic trajectory. If this is the case, this goal can be achieved by means of a feedback control. The stabilization problem to an equilibrium point, a topic first investigated by Cheng, Qi, and Li (2010); Cheng, Qi, Li, and Liu (2011) (see also Li and Sun (2012) and Li, Wang, and Liu (2011) for recent contributions about the stability and stabilizability problems for BCNs and BNs with impulsive effects), follows as a special case.

Notation. Given two nonnegative integers k, n, with $k \leq n$, by the symbol $[k, n]$ we denote the set of integers $\{k, k+1, \dots, n\}$.

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We consider Boolean vectors and matrices, taking values in $\mathcal{B} = \{0, 1\}$, with the usual logical operations (And $\wedge$, Or $\vee$, Negation $^-$). δ_k^i denotes the ith canonical vector of size k, $\mathcal{L}_k$ the set of all k-dimensional canonical vectors, and $\mathcal{L}_{k\times n} \subset \mathcal{B}^{k\times n}$ the set of all $k \times n$ matrices whose columns are canonical vectors of size k. Any matrix $L \in \mathcal{L}_{k\times n}$ can be represented as a vector whose entries are canonical vectors in $\mathcal{L}_k$, namely $L = \begin{bmatrix}\delta_k^{i_1} & \delta_k^{i_2} & \cdots & \delta_k^{i_n}\end{bmatrix}$, for suitable indices $i_1, i_2, \ldots, i_n \in [1, k]$. $[A]_{\ell j}$ is the (ℓ, j)th entry of the matrix A. A *permutation matrix* P is a nonsingular square matrix in $\mathcal{L}_{k\times k}$. In particular, a matrix

$$P = C = \begin{bmatrix}\delta_k^2 & \delta_k^3 & \cdots & \delta_k^k & \delta_k^1\end{bmatrix} \tag{1}$$

is a $k \times k$ *cyclic (permutation) matrix*. Given a matrix $L \in \mathcal{B}^{k\times k}$ (in particular, $L \in \mathcal{L}_{k\times k}$), we associate with it (Brualdi & Ryser, 1991) a *digraph* $\mathcal{D}(L)$, with vertices $1, \ldots, k$. There is an arc (j, ℓ) from j to ℓ if and only if the $[L]_{\ell j} = 1$. A sequence $j_1 \to j_2 \to \cdots \to j_r \to j_{r+1}$ in $\mathcal{D}(L)$ is a *path* of length r from j_1 to j_{r+1} provided that $(j_1, j_2), \ldots, (j_r, j_{r+1})$ are arcs of $\mathcal{D}(L)$. A closed path is called a *cycle*. In particular, a cycle γ with no repeated vertices is called *elementary*, and its length $|\gamma|$ coincides with the number of (distinct) vertices appearing in it. Note that a $k \times k$ cyclic matrix has a digraph that consists of one elementary cycle with length k.

There is a bijective correspondence between Boolean variables $X \in \mathcal{B}$ and vectors $\mathbf{x} \in \mathcal{L}_2$, defined by the relationship

$$\mathbf{x} = \begin{bmatrix} X \\ \overline{X} \end{bmatrix}.$$

We introduce the *(left) semi-tensor product* $\ltimes$ between matrices (and hence, in particular, vectors) as follows (Cheng, Qi, Li, 2011; Laschov & Margaliot, 2012): given $L_1 \in \mathbb{R}^{r_1\times c_1}$ and $L_2 \in \mathbb{R}^{r_2\times c_2}$ (in particular, $L_1 \in \mathcal{L}_{r_1\times c_1}$ and $L_2 \in \mathcal{L}_{r_2\times c_2}$), we set

$$L_1 \ltimes L_2 := (L_1 \otimes I_{T/c_1})(L_2 \otimes I_{T/r_2}), \quad T := \text{l.c.m.}\{c_1, r_2\}.$$

The semi-tensor product represents an extension of the standard matrix product, by this meaning that if $c_1 = r_2$, then $L_1 \ltimes L_2 = L_1L_2$. Note that if $\mathbf{x}_1 \in \mathcal{L}_{r_1}$ and $\mathbf{x}_2 \in \mathcal{L}_{r_2}$, then $\mathbf{x}_1 \ltimes \mathbf{x}_2 \in \mathcal{L}_{r_1r_2}$. By resorting to the semi-tensor product, we can extend the previous correspondence to a bijective correspondence (Cheng, Qi, Li, 2011) between $\mathcal{B}^n$ and $\mathcal{L}_{2^n}$. This is possible in the following way: given $X = \begin{bmatrix}X_1 & X_2 & \cdots & X_n\end{bmatrix}^\top \in \mathcal{B}^n$ set

$$\mathbf{x} := \begin{bmatrix} X_1 \\ \overline{X}_1 \end{bmatrix} \ltimes \begin{bmatrix} X_2 \\ \overline{X}_2 \end{bmatrix} \ltimes \cdots \ltimes \begin{bmatrix} X_n \\ \overline{X}_n \end{bmatrix}.$$

This amounts to saying that

$$\mathbf{x} = \begin{bmatrix} X_1X_2\cdots X_{n-1}X_n \\ X_1X_2\cdots X_{n-1}\overline{X}_n \\ X_1X_2\cdots \overline{X}_{n-1}X_n \\ \vdots \\ \overline{X}_1\overline{X}_2\cdots\overline{X}_{n-1}\overline{X}_n \end{bmatrix}.$$

2. Limit cycles of a Boolean Network

A *Boolean Network* (BN) is described by the following equation

$$X(t + 1) = f(X(t)), \quad t \in \mathbb{Z}_+, \tag{2}$$

where $X(t)$ denotes the n-dimensional state variable at time t, taking values in $\mathcal{B}^n$. f is a (logic) function, namely a map $f : \mathcal{B}^n \to \mathcal{B}^n$. Upon representing the state vector $X(t)$ by means of its equivalent $\mathbf{x}(t)$ in $\mathcal{L}_{2^n}$, the BN (2) can be described (Cheng, Qi, Li, 2011) as

$$\mathbf{x}(t + 1) = L \ltimes \mathbf{x}(t) = L\mathbf{x}(t), \tag{3}$$

where $L \in \mathcal{L}_{2^n\times 2^n}$ is a matrix whose columns are canonical vectors of size 2^n.

Definition 1. An ordered sequence of distinct vectors $(\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_k})$ is a *limit cycle* $\mathcal{C}$ of the BN (3) if $\mathbf{x}(0) = \delta_{2^n}^{i_\ell}$ for some $\ell \in [1, k]$ ensures that the corresponding state trajectory $\mathbf{x}(t)$ is periodic of period k and, for every $t \in \mathbb{Z}_+$, $\mathbf{x}(t) = \delta_{2^n}^{i_j}$, where $j \in [1, k]$ and $j \equiv (t + \ell) \bmod k$. A limit cycle of unitary length is an *equilibrium point* of the BN.

Definition 2. A *limit cycle* $\mathcal{C}$ of the BN (3) is *globally attractive* if for every $\mathbf{x}(0) \in \mathcal{L}_{2^n}$ there exists $\tau \in \mathbb{Z}_+$ such that $\mathbf{x}(t)$ is a state of $\mathcal{C}$ for every $t \in \mathbb{Z}_+$, $t \geq \tau$.

Clearly, a BN has a globally attractive limit cycle if and only if all its state trajectories converge in a finite number of steps to the same periodic trajectory. In order to provide a characterization of globally attractive limit cycles, we introduce the following result.

Proposition 1 (Fornasini & Valcher, in press)**.** *Given a BN (3), there exist $r \in \mathbb{N}$ and a permutation matrix $P \in \mathcal{L}_{2^n\times 2^n}$ such that*

$$P^\top LP = \text{blockdiag}\{D_1, D_2, \ldots, D_r\}, \tag{4}$$

$$\text{with } D_i = \begin{bmatrix} N_i & 0 \\ T_i & C_i \end{bmatrix} \in \mathcal{L}_{n_i\times n_i}, \tag{5}$$

where N_i is a $(n_i - k_i) \times (n_i - k_i)$ square nilpotent matrix, and C_i is a $k_i \times k_i$ cyclic permutation matrix.

The permutation matrix P corresponds to a so called *change of basis* in the vector space of the logic functions of $x_1, x_2, \ldots, x_n$ (Cheng, Qi, Li, 2011). The previous proposition relates a number of properties of the BN to the algebraic structure of L: in the general case, a BN has r limit cycles. Every limit cycle (in particular, every equilibrium point) has a *domain of attraction*, namely a set of initial conditions $\mathbf{x}(0)$ that originate trajectories entering the cycle in a finite number of steps. The block structure of $P^\top LP$ clarifies the domain of attraction of each limit cycle. Finally, the number $\tau_r := \max_{i\in[1,r]}(n_i - k_i)$ represents an upper bound on the *transient time*, namely on the maximum number of steps after which $\mathbf{x}(t)$ steadily belongs to a limit cycle. Note that after τ_r steps, every trajectory is periodic with (not necessarily minimum) period $\tau_p :=$ l.c.m.$\{k_i, i \in [1, r]\}$.

The previous comments immediately lead to the following characterization.

Proposition 2. *Given a BN (3), an ordered set of distinct canonical vectors $\mathcal{C} = (\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_k})$ is a globally attractive limit cycle of the BN if and only if there exists a permutation matrix $P \in \mathcal{L}_{2^n\times 2^n}$ such that $P^\top LP$ can be described as in (4)–(5) for $r = 1$, with C_1 a cyclic permutation matrix of size k and, possibly upon a circular permutation of the indices i_ℓ, $P^\top\delta_{2^n}^{i_\ell} = \delta_{2^n}^{2^n-k+\ell}$, for every $\ell \in [1, k]$.*

By Proposition 1, the *characteristic polynomial* of the matrix L takes the form

$$\Delta_L(z) := \det(zI_{2^n} - L) = \left(z^{2^n - \sum_{i=1}^r k_i}\right)\prod_{i=1}^r (z^{k_i} - 1).$$

Consequently, we have the following corollary.

Corollary 1. *A BN (3) has a globally attractive limit cycle (of length k) if and only if $\Delta_L(z) = z^{2^n-k}(z^k - 1)$.*