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Internal structure of coalitions in competitive and altruistic graphical coalitional games☆

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ABSTRACT

This paper introduces a certain graphical coalitional game where the internal topology of the coalition depends on a prescribed communication graph structure among the agents. The game Value Function is required to satisfy four Axioms of Value. These axioms make it possible to provide a refined study of coalition structures on graphs by defining a formal graphical game and by assigning a Positional Advantage, based on the Shapley value, to each agent in a coalition based on its connectivity properties within the graph. Using the Axioms of Value the graphical coalitional game can be shown to satisfy properties such as convexity, fairness, cohesiveness, and full cooperativeness. Three measures of the contributions of agents to a coalition are introduced: marginal contribution, competitive contribution, and altruistic contribution. The mathematical framework given here is used to establish results regarding the dependence of these three types of contributions on the graph topology, and changes in these contributions due to changes in graph topology. Based on these different contributions, three online sequential decision games are defined on top of the graphical coalitional game, and the stable graphs under each of these sequential decision games are studied. It is shown that the stable graphs under the objective of maximizing the marginal contribution are any connected graph. The stable graphs under the objective of maximizing the competitive contribution are the complete graph. The stable graphs under the objective of maximizing the altruistic contribution are any tree.

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1. Introduction

The objective of this paper is to provide a refined study of the internal structure of coalitions. This is accomplished by defining a graphical coalitional game (GCG) wherein the Value Function is required to satisfy four formal axioms. The allocation of value is made by using the Shapley value; this makes the game fair in Myerson's sense (Myerson, 1977). The GCG with Axioms of Value satisfies all of the desirable attributes of general network model given in Jackson and Wolinsky (1996). Furthermore, the Axioms of Value make it possible to study the effects of competition and altruism in the formation of coalitions. The stable structures of coalitions on graphs with respect to competitive and altruistic objectives are studied.

Game theory is a mathematical discipline (Neumann, 1928; Neumann & Morgenstern, 1944), with a rich and old history (Dimand & Dimand, 1996; Shubik, 2011; Tzu, 1988); it deals with issues and strategies involving competitions and cooperation between several entities (Peters, 2008; Shubik, 2011). In the scope of game theory these entities are called agents (Peters, 2008; Saad, Han, Debbah, Hjorungnes, & Başar, 2009; Shoham & Brown, 2009).

Game theory is primarily divided into two areas: noncooperative game theory (Başar & Olsder, 1999) and cooperative game theory (Peters, 2008; Shoham & Brown, 2009). In noncooperative game theory the fundamental unit of study is the individual agent, and the theory deals with its performance and strategies in the interaction with other individual agents. By contrast, in cooperative game theory the fundamental unit is the set of agents or coalition. Cooperative game theory deals with the value of the coalition, payoff allocations to individual agents, and the stability of coalitions (Peters, 2008; Shoham & Brown, 2009).

Cooperative games can be divided into three classes: Canonical Coalitional Games, Coalition Formation Games, and Coalitional Graph Games (Başar & Olsder, 1999; Saad et al., 2009). Canonical coalitional games deal with the stabilization of the grand coalition. Methods are sought to allocate the net value of the coalition to individual agents in such a way that agents are encouraged to join the coalition. A fair allocation (Myerson, 1977) that often accomplishes

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this is the Shapley value (Shapley, 1953). Coalition formation games study the structure of the coalition based on gains and costs. Finally, the coalitional graph games deal with the formation and stability of coalitions given an underlying communication graph structure (Baras, Jiang, & Purkayastha, 2008; Saad et al., 2009). In the work of Baras (Baras & Jiang, 2005; Baras et al., 2008) and of Başar (Saad, Han, Başar, Debbah, & Hjorungnes, 0000, 2011; Saad et al., 2009) coalitional graph games are studied with applications to communication networks. Various definitions of value are used in Saad et al. (0000), Saad et al. (2009) and Saad et al. (2011), including the probability of detection, gain of resources of other agents, effective throughput, and packet success rate.

Closely related to the coalitional graph games are online or sequential-in-time decision games. These are games where agents make moves through time to maximize their prescribed objective functions (Shoham & Brown, 2009). These games are defined by specifying the allowed moves, and the objective function the agents seek to maximize. These online sequential decision games model real-life situations where the agents are free to change their alliances as considered suitable by them to obtain their objectives.

In his classic work (Myerson, 1977), Myerson used graph theoretic ideas to analyze cooperation in coalitional graph games. He proposed to restrict the interactions in coalitions based on the underlying communication graph structure. He showed that the unique fair (in his sense) allocation of the net value of the coalition to the agents is given by the Shapley value (Shapley, 1953). In their paper (Jackson & Wolinsky, 1996), Jackson and Wolinsky analyzed the stability of networks when the individual agents can choose to form and maintain the links between them. An agent gains value on connecting to another agent and accrues a cost based on maintaining direct communication links with its neighbors. It is shown that different relations between the parameters of link cost and the propagation of value along a path result in the stability of different structures (Bondy & Murty, 2008; Diestel, 2010), such as complete graph, star graph, etc.

Some models closely related to Jackson and Wolinsky (1996) model have been introduced for various definitions of cost and derived gains. Johnson and Gilles (2000) presents a model where the cost of an edge depends upon the geographic distance. In Armengol and Jackson (2001), the probability of the passage of information is taken as the gain parameter in Jackson and Wolinsky (1996). Jackson and Watts (2002) present a dynamic model based on Jackson and Wolinsky (1996). Currarini and Morelli (2000), and Goyal and Joshi (2006) make allocation to agents based on their demands. In Jackson (2009), social setting gives the derived impact.

Certain other specialized models are also introduced to study economic and social networks in specific situations, including: Free-Trade Networks (Furusawa & Konishi, 2007), Market Sharing Agreements (Belleflamme & Bloch, 2004), A Model of Buyer–Seller Networks (Kranton & Minehart, 2001), Buyer–Seller Networks with Quality Differentiated Products (Wang & Watts, 2006), Network Games (Galeotti, Goyal, Jackson, Vega-Redondo, & Yariv, 2010), Formation of NWs with transfer among agents (Bloch & Jackson, 2007), Co-Author Model (Jackson & Wolinsky, 1996), and allocation assignment rules for network games (Jackson, 2005).

The coalitional model presented in this paper is based on Myerson's work (Myerson, 1977) and provides a framework to assign values to the coalition structures and anonymous agents based on connectivity, resembling the one provided in Jackson and Wolinsky (1996). It is shown in the examples that the framework in this paper allows the differentiation between different coalition structures on graphs in situations when the framework in Jackson and Wolinsky (1996) does not.

The first point of impact of the paper is the definition of a GCG where the Value Function is required to satisfy four formal axioms. These axioms allow the development of a rigorous foundation in terms of lemmas and theorems that tie the coalition structure to graph topology; all these results are based on the axioms and cannot be proven without them. The Shapley value with the Value Function satisfying the four Axioms is interpreted as the worth of an agent in a coalition and is called the Positional Advantage (PA). PA strengthens the definition of the Shapley value and formalizes the notion of well-connectedness in communication graphs. Axioms and PA make it possible to formally prove certain attributes (Jackson & Wolinsky, 1996) and properties of the GCG (Arney & Peterson, 2008; Bondareva, 1963; Myerson, 1977; Shapley, 1967).

The second point of impact is to study three types of contributions of agents within a coalition: the marginal, competitive, and altruistic contributions (Arney & Peterson, 2008). Using the Axioms of Value, dependence of these three contributions on graph topology and changes in topology is rigorously developed. These notions cannot be developed without the Axiom of Value.

The third point of impact is the definition of three online sequential decision games (SDG) based on the marginal, competitive, and altruistic contributions, wherein agents make or break edges to maximize the respective contributions. Using the Axioms of Value, it is shown that these SDGs have different stable coalition structures. These stable structures are inherent properties of the objective functions of the games, not parameter dependent as in Jackson and Wolinsky (1996).

The paper is organized as follows. A GCG with formal Axioms of Value is defined in Section 2. Some practical examples are presented to situate the work and compare it to Jackson and Wolinsky (1996). In Section 3, using the Axioms of Value it is shown that the GCG obey all the attributes in Jackson and Wolinsky (1996) and satisfy certain key coalitional properties. In Section 4 three types of contributions of agents in a coalition are discussed – the marginal, competitive, and altruistic contributions. The dependence of these contributions on topological graph properties is also discussed. In Section 5, three online sequential decision games are defined on top of the GCG. The stable graph structures under each of these SDGs are studied. Simulation results on SDG are presented in Section 6 and shown to support the stable graph structures.

2. Positional Advantage in graphical coalitional games

In this section, a graphical coalitional game (GCG) is defined, where the Value Function is required to satisfy four formal axioms. All the results in this paper require one or more of these axioms and cannot be proven without them. The Shapley value (Shapley, 1953), where the Value Function is required to satisfy the four axioms, is defined as Positional Advantage (PA). PA is a fundamental notion in this paper; it formalizes the notion of well-connectedness within a coalition on a graph topology and captures the worth of an agent. This section starts with essential notions of graph theory (Bondy & Murty, 2008; Diestel, 2010).

2.1. Graph definitions

Consider a graph $G = (V, E)$ with V a finite nonempty set of agents and $E \subseteq [V]^2$ a set of edges. Here $[V]^2$ is the unordered set containing all the subsets of V with two elements. Two agents are interpreted to have an edge between them if and only if they directly communicate with each other. The elements of V are also called vertices or nodes. The number of elements in V is called the order or size of the graph and is denoted as $|G|$, also denoted as N. A simple graph does not contain self-loops and multiple edges. Moreover, all its edges are undirected, connecting two vertices, and do not have any weight associated with them. In this paper simple graphs are considered.

Two vertices with an edge between them are called neighbors of each other. If all the vertices of G are neighbors of each other then G is called a complete graph. A complete graph with N vertices is denoted as K_N. A sequence of distinct vertices $i = i_0, i_1, \ldots, i_M = j$