Automatica 50 (2014) 369–377

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Explicit simplicial discretization of distributed-parameter port-Hamiltonian systems[☆]

Marko Seslija[a,1], Jacquelien M.A. Scherpen[b], Arjan van der Schaft[c]

[a] Kasteelpark Arenberg 10, B-3001 Leuven, Belgium

[b] Institute for Technology, Engineering and Management, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands

[c] Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands

ARTICLE INFO

Article history:
Received 17 August 2012
Received in revised form
22 June 2013
Accepted 11 October 2013
Available online 17 December 2013

Keywords:
Port-Hamiltonian systems
Dirac structures
Distributed-parameter systems
Structure-preserving discretization
Discrete geometry

ABSTRACT

Simplicial Dirac structures as finite analogues of the canonical Stokes–Dirac structure, capturing the topological laws of the system, are defined on simplicial manifolds in terms of primal and dual cochains related by the coboundary operators. These finite-dimensional Dirac structures offer a framework for the formulation of standard input–output finite-dimensional port-Hamiltonian systems that emulate the behavior of distributed-parameter port-Hamiltonian systems. This paper elaborates on the matrix representations of simplicial Dirac structures and the resulting port-Hamiltonian systems on simplicial manifolds. Employing these representations, we consider the existence of structural invariants and demonstrate how they pertain to the energy shaping of port-Hamiltonian systems on simplicial manifolds.

1. Introduction

A wide class of field theories can be treated as port-Hamiltonian systems (Schöberl & Schlacher, 2011; van der Schaft & Maschke, 2002). The Stokes–Dirac structure defined by van der Schaft and Maschke (2002) is an infinite-dimensional Dirac structure which provides a theoretical account that permits the inclusion of varying boundary variables in the boundary problem for partial differential equations. From an interconnection and control viewpoint, such a treatment of boundary conditions is essential for the incorporation of energy exchange through the boundary, since in many applications the interconnection with the environment takes place precisely through the boundary. For numerical integration, simulation and control synthesis, it is of paramount interest to have finite-dimensional approximations that can be interconnected to one another.

Most of the numerical techniques emanating from the field of numerical analysis, however, fail to capture the intrinsic system structures and properties, such as symplecticity, conservation of momenta and energy, as well as differential gauge symmetry. Mixed finite element methods can be constructed in a such a manner that a number of important structural properties are preserved (Bossavit, 1998; Hiptmair, 2002; Hirani, 2003). Most of the efforts have been focused on systems on manifolds without boundary or zero energy flow through the boundary. In Golo, Talasila, van der Schaft, and Maschke (2004) a mixed finite element scheme for structure-preserving discretization of port-Hamiltonian systems was proposed. The construction is clear in a one-dimensional spatial domain, but becomes complicated for higher spatial domains. Furthermore, the geometric content of the discretized variables remains moot, in sense that, for instance, the boundary variables do not genuinely live on the geometric boundary.

Recently in Seslija, van der Schaft, and Scherpen (2012), we suggested a discrete exterior geometry approach to structure-preserving discretization of distributed-parameter port-Hamiltonian systems. The spatial domain in the continuous theory represented by a finite-dimensional smooth manifold is replaced by a homological manifold-like simplicial complex and its circumcentric dual. The smooth differential forms, in discrete setting, are mirrored by cochains on the primal and dual complexes, while the discrete exterior derivative is defined to be the coboundary operator. Discrete analogues of the Stokes–Dirac structure are the so-called simplicial Dirac structures defined on spaces of primal and dual discrete differential forms. These finite-dimensional

[☆] The research reported in this paper was supported by the Netherlands Organization for Scientific Research (NWO). The material in this paper was partially presented at 7th Vienna International Conference on Mathematical Modeling (MathMod 2012), February 15–17, 2012, Vienna, Austria. This paper was recommended for publication in revised form by Associate Editor Nicolas Petit under the direction of Editor Miroslav Krstic.

E-mail addresses: Marko.Seslija@gmail.com (M. Seslija), J.M.A.Scherpen@rug.nl (J.M.A. Scherpen), A.J.van.der.Schaft@rug.nl (A. van der Schaft).

[1] Tel.: +32 0 16 32 86 56; fax: +32 0 16 32 19 70.

0005-1098/$ – see front matter
http://dx.doi.org/10.1016/j.automatica.2013.11.020

Dirac structures offer a natural framework for the formulation of finite-dimensional port-Hamiltonian systems that emulate their infinite-dimensional counterparts. The resulting port-Hamiltonian systems are in the standard *input–output* form, unlike in Golo et al. (2004), where the discretized models are *acausal* (given by a set of differential and algebraic equations). The explicit input–output form obtained by our scheme has the advantage from both numerical and control perspective over the implicit model presented in Golo et al. (2004).

In this paper, we address the issue of matrix representations of simplicial Dirac structures by representing cochains by their coefficient vectors. In this manner, all linear operator from the continuous world can be represented by matrices, including the Hodge star, the coboundary and the trace operator. Firstly, we recall the definition of the Stokes–Dirac structure and port-Hamiltonian systems. In the third section, we define some essential concepts from discrete exterior calculus as developed by Desbrun, Hirani, Leok, and Marsden (2002) and Hirani (2003). In order to allow the inclusion of nonzero boundary conditions on the dual cell complex, in Seslija et al. (2012) we have adapted a definition of the dual boundary operator that leads to a discrete analogue of the integration-by-parts formula, which is a crucial ingredient in establishing simplicial Dirac structures on a primal simplicial complex and its circumcentric dual. We demonstrate how these simplicial Dirac structures relate to the spatially discretized wave equation on a bounded domain and to the telegraph equations on a segment. Towards the end of the paper, we consider the existence of structural invariants, which are crucial for the control by energy shaping.

Goal and contributions. This paper is written with several purposes in mind.

- The essential theoretical results of this paper pertaining to structure-preserving discretization, namely, Sections 3–5 have been already reported in Seslija, Scherpen, and van der Schaft (2011, 2012); Seslija et al. (2012) in an algebraic topology setting. The results in this paper do not lean onto the heavy nomenclature of algebraic topology, but instead emphasize matrix representations, making it more accessible and easier to implement. We demonstrate that a discrete differential modeling approach to consistent discretization of distributed-parameter systems is quite approachable—and, in fact, is often much simpler than its continuous counterpart.
- We aim to render the theoretic foundation of our exposition accessible to control theorists, and the paper as such serves as a segue to the rich literature on the subject.
- Another contribution of this paper is given in Sections 7 and 8. Here we address the existence of dynamical invariants for the obtained spatially discrete systems and look at the energy-Casimir method for energy shaping. We anticipate that this line of research will lead to more elaborate and fruitful control strategies for distributed systems.
- We hope that by the end of the paper it will become clear that the discrete geometry-based approach to modeling is *not* only tied to the discretization of infinite-dimensional systems, but, instead, stands as a potent language for the system and control community.

2. Background of port-Hamiltonian systems

Dirac structures were originally developed by Courant (1990) and Dorfman (1993) as a generalization of symplectic, presymplectic and Poisson structures. Later, Dirac structures were employed as the geometric formalism underpinning generalized interconnected and constrained Hamiltonian systems (van der Schaft, 2000; van der Schaft & Maschke, 2002).

2.1. Dirac structures

Let $\mathcal{X}$ be a manifold and define a pairing on $T\mathcal{X} \oplus T^*\mathcal{X}$ given by

$$\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle = \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle.$$

For a subspace $\mathcal{D}$ of $T\mathcal{X} \oplus T^*\mathcal{X}$, we define the orthogonal complement $\mathcal{D}^\perp$ as the space of all (f_1, e_1) such that $\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle = 0$ for all (f_2, e_2). A *Dirac structure* is then a subbundle $\mathcal{D}$ of $T\mathcal{X} \oplus T^*\mathcal{X}$ which satisfies $\mathcal{D} = \mathcal{D}^\perp$.

The notion of Dirac structures is suitable for the formulation of closed Hamiltonian systems, however, our aim is a treatment of open Hamiltonian systems in such a way that some of the external variables remain *free* port variables. For that reason, let $\mathcal{F}_b$ be a linear vector space of external flows, with the dual space $\mathcal{F}_b^*$ of external efforts. We deal with Dirac structures on the product space $\mathcal{X} \times \mathcal{F}_b$. The pairing on $(T\mathcal{X} \times \mathcal{F}_b) \oplus (T^*\mathcal{X} \times \mathcal{F}_b^*)$ is given by

$$\left\langle\!\!\left\langle \Big((f_1, f_{b,1}), (e_1, e_{b,1}) \Big), \Big((f_2, f_{b,2}), (e_2, e_{b,2}) \Big) \right\rangle\!\!\right\rangle = \langle e_1 | f_2 \rangle + \langle e_{b,1} | f_{b,2} \rangle + \langle e_2 | f_1 \rangle + \langle e_{b,2} | f_{b,1} \rangle. \tag{1}$$

A *generalized Dirac structure* $\mathcal{D}$ is a subbundle of $(T\mathcal{X} \times \mathcal{F}_b) \oplus (T^*\mathcal{X} \times \mathcal{F}_b^*)$ which is maximally isotropic under (1).

Consider a generalized Dirac structure $\mathcal{D}$ on the product space $\mathcal{X} \times \mathcal{F}_b$. Let $H : \mathcal{X} \to \mathbb{R}$ be a Hamiltonian. The *port-Hamiltonian system* corresponding to a 4-tuple $(\mathcal{X}, \mathcal{F}_b, \mathcal{D}, H)$ is defined by a set of smooth time-functions $\{t \mapsto (x(t), f_b(t), e_b(t)) \in \mathcal{X} \times \mathcal{F}_b \times \mathcal{F}_b^* | t \in I \subset \mathbb{R}\}$ satisfying the equation

$$(-\dot{x}(t), f_b(t), \mathrm{d}H(x(t)), e_b(t)) \in \mathcal{D} \quad \text{for } t \in I. \tag{2}$$

The Eq. (2) implies the energy balance $\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) = \langle \mathrm{d}H(x(t)) | \dot{x}(t) \rangle = \langle e_b(t) | f_b(t) \rangle$.

An important class of finite-dimensional port-Hamiltonian systems is given by

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + g(x) e_b$$
$$f_b = g^{\mathrm{T}}(x) \frac{\partial H}{\partial x}, \tag{3}$$

where for clarity we have omitted the argument t, and $J : T^*\mathcal{X} \to T\mathcal{X}$ is a skew-symmetric vector bundle map and $g : \mathcal{F}_b \to T\mathcal{X}$ is the independent input vector field.

In this work, we deal exclusively with Dirac structures on linear spaces, which can be defined as follows. Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces. Given an $f \in \mathcal{F}$ and an $e \in \mathcal{E}$, the pairing will be denoted by $\langle e | f \rangle \in \mathbb{R}$. By symmetrizing the pairing, we obtain a symmetric bilinear form $\langle\langle , \rangle\rangle : \mathcal{F} \times \mathcal{E} \to \mathbb{R}$ naturally given as $\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle = \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle$.

A *constant Dirac structure* is a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that $\mathcal{D} = \mathcal{D}^\perp$, with $\perp$ standing for the orthogonal complement with respect to the bilinear form $\langle\langle , \rangle\rangle$.

2.2. Stokes–Dirac structure

The *Stokes–Dirac structure* is an infinite-dimensional Dirac structure that provides a foundation for the port-Hamiltonian formulation of a class of distributed-parameter systems with boundary energy flow (van der Schaft & Maschke, 2002).

Hereafter, let M be an oriented n-dimensional smooth manifold with a smooth $(n-1)$-dimensional boundary ∂M endowed with the induced orientation, representing the space of spatial variables. Adhering to the familiar ground in this paper, M shall be a bounded Euclidean domain. By $\Omega^k(M)$, $k = 0, 1, \ldots, n$, denote the space of exterior k-forms on M, and by $\Omega^k(\partial M)$, $k = 0, 1, \ldots, n-1$, the space of k-forms on ∂M.