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Sampling theorem for the short-time linear canonical transform and its applications

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ABSTRACT

In this paper, we propose a sampling theorem for the short-time linear canonical transform (STLCT) by means of a generalized Zak transform associated with the linear canonical transform (LCT). The sampling theorem, which states that the signal can be reconstructed from its sampled STLCT, turns out to be a generalization of the conventional sampling theorem for the short-time Fourier transform (STFT). Based on the new sampling theorem, Gabor's signal expansion in the LCT domain is obtained, which can be considered as a generalization of the classical Gabor expansion and the fractional Gabor expansion, and presents a simpler method for reconstructing the signal from its sampled STLCT. The derived bi-orthogonality relation of the generalized Gabor expansion is as simple as that of the classical Gabor expansion, and examples are proposed to verify it. Some potential applications of the linear canonical Gabor spectrum for non-stationary signal processing are also discussed.

1. Introduction

The short-time Fourier transform (STFT), which can reveal the local frequency contents of a signal, has been shown to be very useful in time–frequency analysis [1–3]. It can be considered as the Fourier transform (FT) of the product of a signal $f(t)$ and a conjugated and shifted version of a window function $h(t)$ [1–3], i.e.,

$$S_h(t,\omega)=\int_{-\infty}^{+\infty} f(\tau)h^*(\tau-t)e^{-j\omega\tau}\,\mathrm{d}\tau \tag{1}$$

The sampling theorem for the STFT given in [4,5] is an important result in signal processing theory and application. It shows that the signal can be reconstructed from its sampled STFT by means of the classical Zak transform (ZT) [4–8].

The linear canonical transform (LCT) is a three free parameter class of linear integral transform [9–14]. It was proposed in 1970s [9,10] and includes the FT, the fractional Fourier transform (FRFT), the Fresnel transform (FST), and the scaling operations as its special cases [11–14]. The LCT has found many applications in optics, signal and image processing, and pattern recognition [11–13].

With in-depth research on the LCT, many novel time–frequency analysis tools associated with the LCT are currently derived, such as the generalized Wigner distribution (WDL) [13,15], the generalized ambiguity function (AFL) [13,16], and the unified Wigner distribution and ambiguity function (UWA) [14], which are the generalization of corresponding time–frequency analysis tools associated with the FT and FRFT [12]. The short-time linear canonical transform (STLCT), with a local window function, is of importance in signal and image processing and can be considered as a generalization of the STFT by substituting the FT kernel with the LCT kernel directly [17,18]. It can reveal the local LCT-frequency contents of a signal due to its local kernel [17,18].

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Recently, some properties and applications of the STLCT have been investigated in detail. To be specific, the Paley–Wiener theorems and uncertainty principles for the STLCT are obtained in [17]. In addition, the Poisson summation formula and series expansions for the STLCT are derived in [18]. It also presents three kinds of sampling formulae for the STLCT, which establish the links between the original analog signal and its samples, the original analog signal and the samples of the signal and its first derivative, the original analog signal and the samples of the signal and its generalized Hilbert transform [18]. However, the sampling theorem of the STLCT for reconstructing the signal from its sampled STLCT, which parallels with the conventional result for the STFT, is still unknown.

In this paper, we show that with a generalized ZT associated with the LCT that differs from the one introduced in [8], a sampling theorem for the STLCT is obtained and includes the conventional sampling theorem for the STFT as its special case. The theorem indicates that the signal can be reconstructed from its sampled STLCT. Then, we derive Gabor's signal expansion in the LCT domain based on the new sampling theorem. This gives a simpler method for reconstructing the signal from its sampled STLCT. Finally, some examples are performed to show the rationality of the proposed methods and many potential applications of the derived results are also investigated.

The paper is organized as follows. In Section 2, we review the conventional sampling theorem for the STFT. Section 3 presents a sampling theorem for the STLCT based on a generalized ZT associated with the LCT. The application of the new sampling theorem in Gabor's signal expansion is investigated in Section 4. Section 5 put forward many examples to verify the correctness of the proposed techniques. Section 6 describes some possible applications of the derived results for non-stationary signal processing. Finally, we conclude in Section 7.

2. Sampling theorem for the STFT

In this section, we give a brief review on the conventional sampling theorem for the STFT.

We denote the values of the STFT at the sampling points $(t=mT, \omega=k\Omega)$ with $T\Omega=2\pi$ by s_{mk} [4]. From Eq. (1), we have the relation

$$s_{mk}=S_h(mT,k\Omega)=\int_{-\infty}^{+\infty} f(\tau)h^*(\tau-mT)e^{-jk\Omega\tau}\,\mathrm{d}\tau \tag{2}$$

The ZT of a signal $f(t)$ is defined as [4–8]

$$\tilde{f}(t,\omega)=\sum_m f(t+mT)e^{-jmT\omega} \tag{3}$$

Since the function $\tilde{f}(t,\omega)$ is periodic in the frequency variable ω with period Ω, we get the inverse relationship of Eq. (3) as follows:

$$f(t+mT)=\frac{1}{\Omega}\int_\Omega \tilde{f}(t,\omega)e^{jmT\omega}\,\mathrm{d}\omega \tag{4}$$

where $\int_\Omega \mathrm{d}\omega$ stands for an integration over an interval with size Ω.

With Eqs. (2) and (3), an important relational expression can be derived [4,5]:

$$\hat{s}(t,\omega)=T\tilde{f}(t,\omega)\tilde{h}^*(t,\omega) \tag{5}$$

where

$$\hat{s}(t,\omega)=\sum_m\sum_k s_{mk}e^{-j(mT\omega-k\Omega t)} \tag{6}$$

Based on Eqs. (4)–(6), the sampling theorem for the STFT can be described as [4,5]

$$\begin{aligned} f(t+nT)&=\frac{1}{2\pi}\int_\Omega \frac{\hat{s}(t,\omega)}{\tilde{h}^*(t,\omega)}e^{jnT\omega}\,\mathrm{d}\omega \\ &=\frac{1}{2\pi}\sum_m\sum_k\int_\Omega \frac{s_{mk}e^{jk\Omega t}}{\tilde{h}^*(t,\omega)}e^{j(n-m)T\omega}\,\mathrm{d}\omega \end{aligned} \tag{7}$$

Eq. (7) shows that the signal can be reconstructed from its sampled STFT.

3. Sampling theorem for the STLCT

In this section, we first review the definition of the STLCT. Then, we put forward a generalized ZT associated with the LCT. Finally, we show that with this transformation, a sampling theorem for the STLCT and the corresponding reconstruction formula can be derived and can be regarded as the generalization of the conventional results for the STFT. As a special case, we also obtain the sampling theorem for the short-time fractional Fourier transform (STFRFT).

3.1. The short-time linear canonical transform

The LCT is a kind of unified linear integral transform that popularizes the time–frequency domain to the LCT domain. It was applied to solve the differential equations and analyze the optical systems in the early years. The LCT of a signal $f(t)$ with parameter matrix A is defined as [9–18]

$$F_A(u)=L_A[f](u)=\begin{cases}\int_{-\infty}^{+\infty} f(t)K_A(u,t)\,\mathrm{d}t, & b\neq 0\\ \sqrt{d}e^{j(cd/2)u^2}f(du), & b=0\end{cases} \tag{8}$$

where the LCT kernels $K_A(\bullet,\diamond)$ are given by

$$K_A(\bullet,\diamond)=\frac{1}{\sqrt{j2\pi b}}e^{j[(d/2b)(\bullet)^2-(1/b)(\bullet)(\diamond)+(a/2b)(\diamond)^2]} \tag{9}$$

The parameter matrix $A=(a,b;c,d)$ and the parameters a, b, c, d are real numbers satisfying $ad-bc=1$. The LCT has many important properties, for example the additivity, the reversibility, the time shift and frequency shift properties. The interested reader can find the details in [12,13].

When parameter matrix has the special form $A=(0,1;-1,0)$, the LCT reduces to the FT. Then, the STFT can be generalized to the STLCT by substituting the kernel of FT with the kernel of LCT.

The STLCT of a signal $f(t)$ added with a window function $h(t)$ is defined as [17,18]

$$S_h^A(t,u)=\int_{-\infty}^{+\infty} f(\tau)h^*(\tau-t)K_A(u,\tau)\,\mathrm{d}\tau \tag{10}$$