Automatica 50 (2014) 442–451

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Analysis of quantum linear systems' response to multi-photon states[☆]

Guofeng Zhang[1]

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China
Shenzhen Research Institute, The Hong Kong Polytechnic University, Shenzhen, China

ARTICLE INFO

Article history:
Received 10 May 2013
Received in revised form
29 September 2013
Accepted 10 November 2013
Available online 26 December 2013

Keywords:
Quantum linear systems
Multi-photon states
Tensors
System response

ABSTRACT

The purpose of this paper is to present a mathematical framework for analyzing the response of quantum linear systems driven by multi-photon states. Both the factorizable (namely, no correlation among the photons in the channel) and unfactorizable multi-photon states are treated. Pulse information of a multi-photon input state is represented in terms of tensor, and the response of quantum linear systems to multi-photon input states is characterized by tensor operations. Analytic forms of output correlation functions and output states are derived. The proposed framework is applicable no matter whether the underlying quantum dynamic system is passive or active. Examples from the physics literature are used to illustrate the results presented.

1. Introduction

Analysis of system response to various types of input signals is fundamental to control systems engineering. Step response enables a control engineer to visualize system transient behavior such as rise time, overshoot and settling time; frequency response design methods are among the most powerful methods in classical control theory; response analysis of linear systems initialized in Gaussian states driven by Gaussian input signals is the basis of Kalman filtering and linear quadratic Gaussian (LQG) control.

Over the last two decades, there has been rapid advance in experimental demonstration and theoretical investigation of quantum (namely, non-classical) control systems due to their promising applications in a wide range of areas such as quantum communication, quantum computation, quantum metrology, laser-induced chemical reaction, and nano electronics (Albertini & D'Alessandro, 2003; Altafini, 2007; Bolognani & Ticozzi, 2010; Doherty & Jacobs, 1999; Dong & Petersen, 2010; Gardiner & Zoller, 2000; Gough & James, 2009; Huang, Tarn, & Clark, 1983; Li & Khaneja, 2009; Mirrahimi & van Handel, 2007; Nurdin, James, & Doherty, 2009; Qi, 2013; Stockton, van Handel, & Mabuchi, 2004; Wang & Schirmer, 2010; Yamamoto & Bouten, 2009; Yanagisawa & Kimura, 2003; Zhang & James, 2011; Zhang, Wu, Liu, Li, & Tarn, 2012). Within this program quantum linear systems play a prominent role. Quantum linear systems are characterized by linear quantum stochastic differential equations (linear QSDEs). In quantum optics, linear systems are widely used because they are easy to manipulate and, more importantly, linear dynamics often serve well as good approximation of more general dynamics (Gardiner & Zoller, 2000; Wiseman & Milburn, 2010). Besides their broad applications in quantum optics, linear systems have also found applications in many other quantum-mechanical systems such as opto-mechanical systems (Massel et al., 2011), circuit quantum electrodynamics (circuit QED) systems (Matyas, Jirauschek, Peretti, Lugli, & Csaba, 2011), and atomic ensembles (Stockton et al., 2004). From a signals and systems point of view, quantum linear systems driven by Gaussian input states have been studied extensively, and results like quantum filtering and measurement-based feedback control have been well established (Wiseman & Milburn, 2010).

In addition to Gaussian states there are other types of non-classical states, for example single-photon states and multi-photon states. Such states describe electromagnetic fields with a definite number of photons. Due to their highly non-classical nature and recent hardware advance, there is rapidly growing interest in the generation and engineering (e.g., pulse shaping) of photon states, and it is generally perceived that these photon states hold promising applications in quantum communication, quantum computing, quantum metrology and quantum simulations (Bartley et al., 2012; Cheung, Migdall, & Rastello, 2009; Gheri, Ellinger, Pellizzari, & Zoller, 1998; Gough, James, & Nurdin, 2013; Milburn, 2008; Ou, 2007; Sanaka, Resch, & Zeilinger, 2006). Thus, a new and important problem in the field of quantum control engineering is: How to characterize and engineer interaction between quantum

[☆] This research is supported in part by a National Natural Science Foundation of China (NSFC) grant (No. 61374057) and a Hong Kong RGC grant (No. 531213). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor James Lam under the direction of Editor Ian R. Petersen.

E-mail address: Guofeng.Zhang@polyu.edu.hk.

[1] Tel.: +86 852 2766 6936; fax: +86 852 2764 4382.

0005-1098/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.automatica.2013.12.004

linear systems and photon states? The interaction of quantum linear systems with continuous-mode photon states has recently been studied in the literature, primarily in the physics community. For example, interference phenomena of photons passing through beamsplitters have been studied, see, e.g., Bartley et al. (2012), Ou (2007) and Sanaka et al. (2006). Milburn discussed how to use an optical cavity to manipulate the pulse shape of a single-photon light field (Milburn, 2008). Quantum filtering for systems driven by single-photon fields has been investigated in Gough et al. (2013). Intensities of output fields of quantum systems driven by continuous-mode multi-photon light fields have been studied in Baragiola, Cook, Brańczyk, and Combes (2012). In Zhang and James (2013) the response of quantum linear systems to single-photon states has been studied.

In the analysis of the response of quantum linear systems to single-photon states, matrix presentation is sufficient because two indices are adequate: one for input channels, and the other for output channels. However, this is not the case in the multi-photon setting. In addition to indices for input and output channels, we need another index to count photon numbers in channels. As a result, tensor representation and operation are essential in the multi-photon setting. To be specific, multi-photon state processing by quantum linear systems can be mathematically represented in terms of tensor processing by transfer functions. The key ingredient for such an operation is the following (for the passive case). Let $E(t) = (E^{jk}(t)) \in \mathbb{C}^{m\times m}$ be the transfer function of a quantum linear passive system with m input channels. For each $j = 1, \dots, m$, let $\mathscr{V}_j(t_1, \dots, t_{\ell_j})$ be an ℓ_j-way m-dimensional tensor function that encodes the pulse information of the j-th input channel containing ℓ_j photons. Denote the entries of $\mathscr{V}_j(t_1, \dots, t_{\ell_j})$ by $\mathscr{V}_{j,k_1,\dots,k_{\ell_j}}(t_1, \dots, t_{\ell_j})$. For all given $1 \le r_1, \dots, r_{\ell_j} \le m$, define an ℓ_j-way m-dimensional tensor $\mathscr{W}_j$ with entries given by the following multiple convolution

$$\mathscr{W}_{j,r_1,\dots,r_{\ell_j}}(t_1, \dots, t_{\ell_j}) = \sum_{k_1,\dots,k_{\ell_j}=1}^{m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E^{r_1k_1}(t_1 - \iota_1) \times \cdots \times E^{r_{\ell_j}k_{\ell_j}}(t_{\ell_j} - \iota_{\ell_j})\mathscr{V}_{j,k_1,\dots,k_{\ell_j}}(\iota_1, \dots, \iota_{\ell_j})d\iota_1 \dots d_{\ell_j}.$$

It turns out that the tensors $\mathscr{W}_j$ $(j = 1, \dots, m)$ encode the pulse information of the output field. That is, an ℓ_j-way input tensor is mapped to an ℓ_j-way output tensor by the quantum linear passive system.

The contributions of this paper are threefold. First, the analytic form of the steady-state output state of a quantum linear system driven by a multi-photon input state is derived. When the quantum linear system is a beamsplitter (a static passive device), interesting multi-photon interference phenomena studied in Bartley et al. (2012), Ou (2007), and Sanaka et al. (2006) are re-produced by means of our approach, see Examples 1–3. Second, when the underlying quantum linear system is not passive (e.g., a degenerate parametric amplifier), the steady-state output state with respect to a multi-photon input state is not a multi-photon state. In terms of tensor representation, a more general class of states is defined. Such rigorous mathematical description paves the way for multi-photon state engineering. Third, both the factorizable and unfactorizable multi-photon states are treated in this paper. Here a factorizable multi-photon state is a state for which the photons in a given channel are not correlated, while for an unfactorizable multi-photon state there exists correlation among the photons. This difference cannot occur in the single-photon state case. Thus, the mathematical framework presented here is more general.

The rest of the paper is organized as follows. Preliminary results are presented in Section 2. Specifically, quantum linear systems are briefly reviewed in Section 2.1 with focus on stable inversion and covariance function transfer, in Section 2.2 several types of tensors and their associated operations are introduced. The multi-photon state processing when input states are factorizable in terms of pulse shapes is studied in Section 3. (Here the word "factorizable" means there is no correlation among photons in each specific channel.) Specifically, single-channel and multi-channel multi-photon states are presented in Sections 3.1 and 3.2 respectively, covariance functions and intensities of output fields are studied in Section 3.3, while an analytic form of steady-state output states is derived in Section 3.4. The unfactorizable case is investigated in Section 4. Specifically, unfactorizable multi-channel multi-photon states are defined in Section 4.1, the analytic form of the steady-state output state is presented in Section 4.2 where the underlying system is passive, the active case is studied in Section 4.3. Some concluding remarks are given in Section 5.

Notations. m is the number of input channels, and n is the number of degrees of freedom of a given quantum linear stochastic system. $|\phi\rangle$ denotes the initial state of the system which is always assumed to be vacuum, $|0\rangle$ denotes the vacuum state of free fields. Given a column vector of complex numbers or operators $x = [x_1 \ \cdots \ x_k]^T$ where k is a positive integer, define $x^\# = [x_1^* \ \cdots \ x_k^*]^T$, where the asterisk $*$ indicates complex conjugation or Hilbert space adjoint. Denote $x^\dagger = (x^\#)^T$. Furthermore, define the doubled-up column vector to be $\breve{x} = [x^T \quad (x^\#)^T]^T$. Let I_k be an identity matrix and 0_k a zero square matrix, both of dimension k. Define $J_k = \mathrm{diag}(I_k, -I_k)$ and $\Theta_k = [0 \quad I_k; \quad -I_k \quad 0]$. (The subscript "$k$" is often omitted.) Then for a matrix $X \in \mathbb{C}^{2j\times 2k}$, define $X^\flat := J_k X^\dagger J_j$. $\otimes_c$ denotes the Kronecker product. Given a function $f(t)$ in the time domain, define its two-sided Laplace transform to be $F[s] = \mathscr{L}_b\{f(t)\}(s) := \int_{-\infty}^{\infty} e^{-st} f(t)dt$. Given two constant matrices $U, V \in \mathbb{C}^{r\times k}$, define $\Delta(U, V) = [U \ V; V^\# \ U^\#]$. Similarly, given time-domain matrix functions $E^-(t)$ and $E^+(t)$ of compatible dimensions, define $\Delta(E^-(t), E^+(t)) = [E^-(t) \ E^+(t); E^+(t)^\# \ E^-(t)^\#]$. Given two operators A and B, their commutator is defined to be $[A, B] := AB - BA$. For any integer $r > 1$, we write $\int_r$ for integration in the space $\mathbb{R}^r$. We also write $dt_{1\to r}$ for $dt_1 \cdots dt_r$. Finally, given a column vector a, we use a_j to denote its entries. Given a matrix A, we use A^{jk} to denote its entries. Given a 3-way tensor $\mathscr{A}$ (also called a tensor of order 3), we use $\mathscr{A}_{ijk}$ to denote its entries; we do the similar thing for higher order tensors.

2. Quantum linear systems and tensors

This section records preliminary results necessary for the development of the paper. Quantum linear systems are briefly discussed is Section 2.1. Tensors and their associated operations, the appropriate mathematical language for the describe the interaction of a quantum linear system with multi-photon channels, is introduced in Section 2.2.

2.1. Quantum linear systems

In this subsection we set up the model which is a quantum linear system driven by boson fields (Gardiner & Zoller, 2000; Wiseman & Milburn, 2010; Zhang & James, 2011).

2.1.1. Fields and systems

The triple (S, L, H) provides a compact way for the description of open quantum systems (Gough & James, 2009). Here the self-adjoint operator H is the initial system Hamiltonian, S is a unitary scattering operator, and L is a coupling operator that describes how the system is coupled to its environment. The environment is an m-channel electromagnetic field in free space, represented by a column vector of annihilation operators $b(t) = [b_1(t), \dots, b_m(t)]^T$. Let t_0 be the initial time, namely, the time when the quantum system starts interacting with its environment.