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Brief paper

On minimum-time paths of bounded curvature with position-dependent constraints[☆]

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ABSTRACT

We consider the problem of a particle traveling from an initial configuration to a final configuration (given by a point in the plane along with a prescribed velocity vector) in minimum time with non-homogeneous velocity and with constraints on the minimum turning radius of the particle over multiple regions of the state space. Necessary conditions for optimality of these paths are derived to characterize the nature of optimal paths, both when the particle is inside a region and when it crosses boundaries between neighboring regions. These conditions are used to characterize families of optimal and nonoptimal paths. Among the optimality conditions, we derive a "refraction" law at the boundary of the regions that generalizes the so-called Snell's law of refraction in optics to the case of paths with bounded curvature. Tools employed to deduce our results include recent principles of optimality for hybrid systems. A numerical example is given to demonstrate the derived results.

1. Introduction

1.1. Background

Pontryagin's Maximum Principle (Pontryagin, Boltyanskij, Gamkrelidze, & Mishchenko, 1962) is a very powerful tool to derive necessary conditions for optimality of solutions to a dynamical system. In other words, this principle establishes the existence of an adjoint function with the property that, along optimal system solutions, the Hamiltonian obtained by combining the system dynamics and the cost function associated to the optimal control problem is minimized. In its original form, this principle is applicable to optimal control problems with dynamics governed by differential equations with continuously differentiable right-hand sides.

The shortest path problem between two points with specific tangent directions and bounded maximum curvature has received much attention in the literature. In his pioneering work in Dubins (1957), by means of geometric arguments, Dubins showed that optimal paths to this problem consist of a concatenation of no more than three pieces, each of them describing either a straight line, denoted by $\mathcal{L}$, or a circle, denoted by $\mathcal{C}$ (when the circle is traveled clockwise, we label it as $\mathcal{C}^+$, while when the circle is traveled counter-clockwise, $\mathcal{C}^-$), and are either of type $\mathcal{CCC}$ or $\mathcal{CLC}$, that is, they are among the following six types of paths

$$\mathcal{C}^-\mathcal{C}^+\mathcal{C}^-,\ \mathcal{C}^+\mathcal{C}^-\mathcal{C}^+,\ \mathcal{C}^-\mathcal{L}\mathcal{C}^-,\ \mathcal{C}^+\mathcal{L}\mathcal{C}^+,\ \mathcal{C}^+\mathcal{L}\mathcal{C}^-,\ \mathcal{C}^-\mathcal{L}\mathcal{C}^+ \qquad (1)$$

in addition to any of the subpaths obtained when some of the pieces (but not all) have zero length. More recently, the authors in Boissonnat, Cérézo, and Leblond (1994) recovered Dubins' result by using Pontryagin's Maximum Principle. Further investigations of the properties of optimal paths to this problem and other related applications of Pontryagin's Maximum Principle include Balkcom and Mason (2002), Chitsaz, LaValle, Balkcom, and Mason (2006) and Shkel and Lumelsky (2001) to just list a few.

1.2. Contributions

We consider the minimum-time problem of having time-parametrized paths with bounded curvature for a particle, which, as in the problem by Dubins, travels from a given initial point to a final point with specified velocity vectors, but with non-homogeneous traveling speeds and curvature constraints: the velocity of the particle and the minimum turning radius are possibly different at certain regions of the state space. (Note that since the velocity of the particle in the problem by Dubins is constant, the minimum-length and minimum-time problems are equivalent; while the problem with different velocities and curvature

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constraints is most interesting for the minimum-time case.) Such a heterogeneity arises in several robotic motion planning problems across environments with obstacles, different terrains properties, and other topological constraints. Current results for optimal control under heterogeneity, which include those in Alexander and Rowe (1990) and Rowe and Alexander (2000), are limited to particles describing straight paths. Furthermore, optimal control problems exhibiting such discontinuous/impulsive behavior cannot be solved using the classical Pontryagin's Maximum Principle. Extensions of this principle to systems with discontinuous right-hand side appeared in Sussmann (1997) while extensions to hybrid systems include Shaikh and Caines (2007) and Sussmann (1999a). These principles establish the existence of an adjoint function which, in addition to conditions that parallel the necessary optimality conditions in the principle by Pontryagin, satisfies certain conditions at times of discontinuous/jumping behavior. The applicability of these principles to relevant problems have been highlighted in D'Apice, Garavello, Manzo, and Piccoli (2003), Piccoli (1999) and Sussmann (1999a). These will be the key tools in deriving the results in this paper.

Building from preliminary results in Sanfelice and Frazzoli (2008) and exploiting recent principles of optimality for hybrid systems, we establish necessary conditions for optimality of paths of particles with bounded curvature traveling across a state space that is partitioned into multiple regions, each with a different velocity and minimum turning radius. Necessary conditions for optimality of these paths are derived to characterize the nature of optimal paths, both when the particle is inside a region and when it crosses boundaries between neighboring regions. A "refraction" law at the boundary of the regions that generalizes the so-called Snell's law of refraction in optics to the case of paths with bounded curvature is also derived. The optimal control problem with a "refraction" law at the boundary can be viewed as an extension of optimal control problems in which the terminal time is governed by a stopping constraint, as considered in Lin, Loxton, Teo, and Wu (2011, 2012). The necessary conditions we derived also provide a novel alternative to optimizing a switched system without directly optimizing the switching times as decision variables, as is commonly done in a vast majority of papers dealing with switched system optimization, e.g. Jiang, Teo, Loxton, and Duan (2012) and Wu and Teo (2006). Applications of these results include optimal motion planning of autonomous vehicles in environments with obstacles, different terrains properties, and other topological constraints. Strategies that steer autonomous vehicles across heterogeneous terrain using Snell's law of refraction have already been recognized in the literature and applied to point-mass vehicles; see, e.g., Alexander and Rowe (1990) and Rowe and Alexander (2000), and more recently, Kwok and Martinez (2010). Our results extend those to the case of autonomous vehicles with Dubins dynamics, consider the case when the state space is partitioned into finitely many regions, and allow for the velocity of travel and minimum turning radius to change in each region. The results are demonstrated numerically using the software package GPOPS (Rao et al., 2010). An extended version of this paper is available at Sanfelice, Yong, and Frazzoli (2013).

The organization of the paper is as follows. Section 2 states the problem of interest and outlines the solution approach. Section 3 presents the main results: necessary conditions for optimality of paths, refraction law at the boundary of the regions, and characterization of families of optimal and nonoptimal paths. A numerical example is given in Section 4.

1.3. Notation

We use the following notation throughout the paper. $\mathbb{R}^n$ denotes n-dimensional Euclidean space. $\mathbb{R}$ denotes the real numbers. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. Given $k \in \mathbb{N}$, $\mathbb{N}_{\leq k}$ denotes $\{0, 1, \ldots, k\}$ and, if $k > 0$, $\mathbb{N}_{<k}$ denotes $\{0, 1, \ldots, k-1\}$. Given a set S, $\overline{S}$ denotes its closure, S° denotes its interior, and ∂S denotes its boundary. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given vectors x and y, at times, we write $[x^\top, y^\top]^\top$ with the shorthand notation (x, y). Given a function f, its domain is denoted by $\operatorname{dom} f$. Given $\overline{u}_i > 0$ defining $U_i := [-\overline{u}_i, \overline{u}_i]$, $\mathcal{U}_i$ denotes the set of all piecewise-continuous functions u from subsets of $\mathbb{R}_{\geq 0}$ to U_i. The inner product between vectors u and v is denoted by $\langle u, v \rangle$. A unit vector with angle θ is denoted by $\angle\theta$.

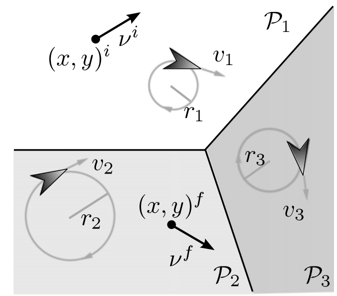

Fig. 1. Initial point $(x, y)^i$ and final point $(x, y)^f$ with given velocity vectors on regions $\mathcal{P}_1$ and $\mathcal{P}_2$. The minimum turning radius in region $\mathcal{P}_1$ is smaller than the one in region $\mathcal{P}_3$, which is smaller than the one in region $\mathcal{P}_2$ as denoted by the depicted paths with minimum turning radius.

2. Problem statement and solution approach

We are interested in deriving necessary conditions for a path $\mathcal{X}$ describing the motion of a particle, which starts and ends at pre-established points with particular velocity vectors, through regions $\mathcal{P}_q$ with different constant velocity v_q and minimum turning radius r_q. The dynamics of a particle with position $(x, y) \in \mathbb{R}^2$ and orientation $\theta \in \mathbb{R}$ (with respect to the vertical axis) are given by

$$\dot{x} = v_q \sin\theta, \qquad \dot{y} = v_q \cos\theta, \qquad \dot{\theta} = u, \tag{2}$$

where $u \in U_q$ is the angular velocity input and satisfies $|u| \leq \overline{u}_q := \frac{v_q}{r_q}$. The velocity vector of the particle is given by the vector $\left[v_q \sin\theta, v_q \cos\theta\right]^\top$. More precisely, we are interested in the following problem:

Problem 1. Given a connected set $\mathcal{P} \subset \mathbb{R}^2$, N disjoint polytopes $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N$, subsets of $\mathcal{P}$, with nonempty interior and such that $\mathcal{P} = \bigcup_{q \in \{1,2,\ldots,N\}} \mathcal{P}_q$, determine necessary conditions on the minimum-time path $\mathcal{X} \subset \mathcal{P}$ of a particle starting at a point $(x, y)^i \in \mathcal{P}^\circ_{q^i}$, $q^i \in \{1, 2, \ldots, N\}$, with initial velocity vector ν^i, traveling according to (2), and ending at a point $(x, y)^f \in \mathcal{P}^\circ_{q^f}$, $q^f \in \{1, 2, \ldots, N\}$, with final velocity vector ν^f, where, for each $q \in \{1, 2, \ldots, N\}$, $v_q > 0$ and $r_q > 0$ are the velocity of travel and the minimum turning radius in $\mathcal{P}_q$, respectively. △

In addition to Problem 1, we consider the special case when the angular velocity constraints on neighboring regions have common bounds $\overline{u}_q$. We refer to the resulting problem as Problem 1⋆.

Fig. 1 depicts the general scenario in Problem 1. Neighboring regions are such that either their velocity of travel, their minimum turning radius, or both are different from each other. In this way, the number of regions with different characteristics is irreducible.

Our approach to derive a solution to Problem 1 is as follows. Given a continuously differentiable curve $\mathcal{X} \subset \mathcal{P}$ defining the path of a particle starting at a point $(x, y)^i \in \mathcal{P}^\circ_{q^i}$, $q^i \in \{1, 2, \ldots, N\}$, with