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Brief paper

Non-existence of finite-time stable equilibria in fractional-order nonlinear systems☆

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ABSTRACT

We note that in the literature it is often taken for granted that for fractional-order system without delays, whenever the system trajectory reaches the equilibrium, it will stay there. In fact, this is the well-known phenomenon of finite-time stability. However, in this paper, we will prove that for fractional-order nonlinear system described by Caputo's or Riemann–Liouville's definition, any equilibrium cannot be finite-time stable as long as the continuous solution corresponding to the initial value problem globally exists. In addition, some examples of stability analysis are revisited and linear Lyapunov function is used to prove the asymptotic stability of positive fractional-order nonlinear systems.

1. Introduction

Fractional-order dynamic systems have received increasing attention in recent years due to their broad range of applications in viscoelastic materials (Bagley & Torvik, 1986), control engineering (Oustaloup, Mathieu, & Lanusse, 1995; Podlubny, 1999b), fractional electrical circuits (Nakagawa & Sorimachi, 1992), and chaotic systems (Lu & Chen, 2006). A distinguished feature of fractional-order systems is their memory effects, which can be utilized to characterize some physical phenomena or complex systems more precisely.

Stability analysis is a basic problem in control theory. The stability region for linear time-invariant fractional-order system is firstly given in Matignon (1996). Then the LMI characterizations for the stability region with fractional order $\alpha \in (1, 2)$ and $\alpha \in (0, 1)$ are presented in Chilali, Gahinet, and Apkarian (1999) and Farges, Moze, and Sabatier (2010), respectively. The stability region for fractional-order system with $\alpha \in (0, 1)$ is non-convex and hence the LMI characterization is more difficult. There are also some results on the stability analysis of fractional-order nonlinear systems, one can see Li, Chen, and Podlubny (2009), Li, Chen, and Podlubny (2010), Trigeassou, Maamri, Sabatier, and Oustaloup (2011) and references therein. However, as discussed in Sabatier, Moze, and Farges (2010), the stability theory for fractional-order nonlinear systems is still not well developed and requires further investigation. As commented in Trigeassou et al. (2011), it may not be realistic to use quadratic Lyapunov functions in the stability analysis of fractional-order systems. One can see how difficult it is to use quadratic Lyapunov function to analyze the stability in Example 14 Li et al. (2009), where the simple scalar fractional differential equation ${}_0^C D_t^\alpha x(t) = -x^3(t)$ is considered. In this paper we also revisit this example and point out that for a class of positive fractional-order systems, linear Lyapunov functions can be used to discuss stability.

Fractional-order systems may have some different properties from the classical integer-order systems. For instance, fractional-order systems described by Caputo's definition cannot produce exact periodic solutions (Tavazoei & Haeri, 2009) and the fractional derivative of a periodic function cannot be a periodic function with the same period (Tavazoei, 2010). Although there is a minor flaw in the proof in Tavazoei and Haeri (2009), the results are confirmed in Tavazoei and Haeri (2012). Since periodic orbits are steady state solutions, one can imagine that fractional-order systems cannot produce steady state solutions in general. Therefore, it is natural to guess that finite-time stable trajectories are also impossible for fractional-order systems. However, it seems that in the literature, it is always taken for granted that whenever the system trajectory hits the equilibrium, it will then remain there forever.

In this paper, we first give a proof that the phenomenon of finite-time stability will never happen in fractional-order systems with either Caputo or Riemann–Liouville derivative. As

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a typical example, we point out that finite-time synchronization of fractional-order chaotic systems via terminal sliding mode control proposed in Aghababa (2012) is not possible. In addition, some examples of stability analysis in the previous literature are revisited and linear Lyapunov functions are used to discuss the stability of positive fractional-order nonlinear systems.

2. Preliminaries

In this section, some basic notions and properties for fractional calculus and fractional differential equations are recalled. For further details, one can refer to Podlubny (1999a). Without loss of generality, it is assumed that the lower limit of the fractional integrals and derivatives is 0 throughout this paper.

The fractional integral with order α is defined as

$$ {}_0D_t^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau $$

where $\Gamma(\cdot)$ is the Gamma function. There are different definitions for fractional-order derivatives among which Caputo derivative is the most frequently used in control engineering. In the following, it is always assumed that $\alpha\in(0,1)$. The αth Caputo derivative of function $f(t)$ is defined by

$$ {}_0^CD_t^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}f'(\tau)\mathrm{d}\tau $$

where f' is the first order derivative of function f. Another definition of fractional-order derivative is Riemann–Liouville derivative which is defined as

$$ {}_0D_t^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\tau)^{-\alpha}f(\tau)\mathrm{d}\tau. $$

The relationship between these two definitions is given by

$$ {}_0^CD_t^{\alpha}f(t)={}_0D_t^{\alpha}f(t)-\frac{f(0)}{\Gamma(1-\alpha)}t^{-\alpha}. $$

In particular, if $f(0)=0$, then we have ${}_0^CD_t^{\alpha}f(t)={}_0D_t^{\alpha}f(t)$.

A function frequently used in the solutions of fractional-order systems is the Mittag-Leffler function which is defined as

$$ E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}. $$

The Mittag-Leffler function with two parameters has the following form:

$$ E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)} $$

where $\alpha>0$ and $\beta>0$. Note that $E_{\alpha,1}(z)=E_\alpha(z)$ and $E_{1,1}(z)=e^z$. The Beta function will also be used in the sequel. In this paper, we use the following equivalent definition of the Beta function (Olver, Lozier, Boisvert, & Clark, 2010):

$$ B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t=\int_0^{+\infty}\frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t $$

for all $x>0, y>0$. In the following, we will introduce the Laplace transform of the Caputo fractional derivative. Let $\mathscr{L}$ denote the Laplace transform of a function. It follows from the definition of Laplace transform $F(s)=\mathscr{L}\{f(t)\}=\int_0^{+\infty}e^{-st}f(t)\mathrm{d}t$ that

$$ \mathscr{L}\{{}_0^CD_t^{\alpha}f(t)\}=s^\alpha F(s)-s^{\alpha-1}f(0). $$

The Laplace transform of Mittag-Leffler function with two parameters is given as

$$ \mathscr{L}\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\}=\frac{s^{\alpha-\beta}}{s^\alpha+\lambda},\quad \left(\mathrm{Re}(s)>|\lambda|^{\frac{1}{\alpha}}\right). $$

3. Main results

3.1. Caputo fractional derivative case

In what follows, we consider a general non-autonomous fractional-order nonlinear system described by Caputo's definition:

$$ {}_0^CD_t^{\alpha_i}x_i(t)=f_i(t,x(t)),\quad i=1,2,\ldots,n \tag{1} $$

where $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T\in\mathbb{R}^n$ and $0<\alpha_i<1,\ i=1,2,\ldots,n$ denote the pseudo-state vector and the fractional order of system (1), respectively; $f_i(t,x),\ i=1,2,\ldots,n$ are nonlinear continuous functions.

As is well known, system (1) is equivalent to a Volterra integral equation (Diethelm & Ford, 2002):

$$ x_i(t)=x_i(0)+\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-\tau)^{\alpha_i-1}f_i(\tau,x(\tau))\mathrm{d}\tau,\quad i=1,2,\ldots,n. \tag{2} $$

There are many results on the existence and uniqueness of solutions for the initial value problem of system (1), see for example Daftardar-Gejji and Jafari (2007) and references therein. In this paper, instead of imposing certain sufficient conditions to guarantee the existence of solutions, let us just assume that system (1) with a given initial condition $x(0)=x_0$ has a continuous solution $x(t)$ on $[0,+\infty)$.

Remark 1. From (2), it is clear that the trajectory of system (1) is well-defined for $t>0$ once $x(0)$ is given, thus in this case $x(t)$ is fully determined by $x(0)$ in the usual sense. The trajectory of system (1) does not require information of the semi-infinite time interval $x(\tau)$ $(-\infty<\tau\le 0)$.

A set $\mathscr{P}$ is called an invariant set with respect to system (1) if for any initial value $x(0)\in\mathscr{P}$, the trajectory $x(t)\in\mathscr{P}$ for $t\ge 0$. System (1) is called a positive system if $\overline{\mathbb{R}}_+^n$ is an invariant set (Kaczorek, 2011).

The constant vector $x^*=(x_1^*,x_2^*,\ldots,x_n^*)^T$ is an equilibrium of system (1) if and only if ${}_0^CD_t^{\alpha_i}x_i^*=f_i(t,x^*)$ for $t\ge 0$ and $i=1,2,\ldots,n$ (Li et al., 2009). Since the Caputo derivative of a constant is 0, the equilibria of system (1) are the points satisfying $f_i(t,x^*)=0$ for $t\ge 0$ and $i=1,2,\ldots,n$.

In this paper, we follow the classical definition of finite-time stability. The equilibrium $x=x^*$ of system (1) is said to be (locally) finite-time stable if it is stable and for the trajectory $x(t)$ starting from x_0 located in a neighborhood of x^*, there exists a time instant $T>0$, such that $x(t)=x^*$ for all $t\ge T$. Obviously, finite-time stability can be regarded as a special case of asymptotic stability. In this section, we will analytically prove that any equilibrium of system (1) cannot be finite-time stable. Before moving on, the following lemma is needed.

Lemma 2. *Given $\alpha\in(0,1)$ and assume that $g(t)$ is defined on $\mathbb{R}$ with $g(t)=0$ for $t\notin[0,T)$. Further assume that $g(t)$ is continuous on $(0,T]$ and is Lebesgue integrable on $[0,T]$, then*

$$ \int_0^T\frac{g(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau=0,\quad \forall t>T \tag{3} $$

implies that $g(t)=0$ for $t\in(0,T]$.

Proof. Note that (3) implies that $t^\beta\int_0^T\frac{g(\tau)}{(t+T-\tau)^\alpha}\mathrm{d}\tau=0$ for all $t>0$ and $\beta\in(-1,\alpha-1)$, hence

$$ \int_0^{+\infty}t^\beta\int_0^T\frac{g(\tau)}{(t+T-\tau)^\alpha}\mathrm{d}\tau\mathrm{d}t=0. $$