ARTICLE IN PRESS

Signal Processing ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Pole-zero placement algorithm for the design of digital filters with fractional-order rolloff

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ARTICLE INFO

Article history:
Received 2 December 2013
Received in revised form
17 April 2014
Accepted 12 May 2014

Keywords:
Digital filters
Discretization
Fractional calculus
Fractional-order systems
Pole-zero placement
Rolloff

ABSTRACT

Digital and analog filters are often used in the modeling of real-world systems. Many practical systems are better modeled by filters with fractional order rolloff. Finite integer-order transfer functions cannot represent fractional-order systems exactly. They can, however, approximate fractional-order systems, with the approximation quality depending on the order of the transfer functions and the methods used to design them. Several analog design techniques have been developed to realize and improve such approximations. To date, digital design techniques have been largely restricted to discretizations of existing analog solutions. This paper presents a novel approach to designing digital lowpass filters with fractional-order rolloff directly in the discrete domain through pole-zero placement. Filters designed using the proposed iterative technique are stable, have precisely definable cutoff frequencies, and do not suffer from the variations that can arise from transforming an existing analog design. The proposed technique is shown to outperform certain existing analog and digital design methods, both subjectively and by objective measures. To complement the proposed approach, a parameter-estimation routine is also introduced to alleviate some of the associated computational burden by reducing its reliance on iterative methods.

1. Introduction

Modeling real-world systems as analog or digital filters with specific transfer function characteristics has been a popular subject of research for years. Often complex systems can be reduced to or modeled by some combination of much more simple filter stages. Historically, these filters have been defined by finite, integer-order transfer functions. Such filters have an ideal transition band slope (*rolloff*) of $\pm 6n$ decibels per octave (dB/oct) or $\pm 20n$ dB per decade (dB/dec), with integer n. Yet the real world is rarely ideal, and many systems would be more accurately modeled with *fractional-order* rolloff filters. Fractional-order rolloff filters are required in such applications as underwater acoustics (to model ambient sea noise [1]) and speech intelligibility (to model the long-term spectrum of human speech [2]). Fractional integrators and differentiators have also been studied extensively.

The concept of extending integer-order mathematics to their fractional-order generalizations is not new. Fractional calculus, the study of generalizing integration and differentiation to fractional orders, dates back more than three centuries [3]. However, early research into fractional calculus was impossible due to the computational complexity involved. By the latter half of the 20th century, computational capability had advanced such that practical

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http://dx.doi.org/10.1016/j.sigpro.2014.05.007
0165-1684/

applications of fractional calculus could be investigated. Since exact fractional-order representations of systems would generally require systems of infinite order, much early research was devoted to finding finite-order approximations. An excellent review of these methods was provided by Vinagre et al. [4], including some of the more significant methods developed by Carlson [5], Roy [6], Oustaloup [7], Charef [8], and Matsuda [9].

The earliest practical applications of fractional-order systems were in analog circuitry. Several early papers, and even a few more recent ones, were devoted to this topic [6,8,10–12]. Much research has also been conducted into using fractional calculus in control theory to describe plants and controllers of fractional order [7,13–18]. Since highpass and lowpass filters can be viewed as band-limited differentiators and integrators, some researchers have investigated using fractional calculus concepts in signal processing and filter design [19–26].

Since early work with fractional-order systems was focused on analog theory and implementations, all the finite-order approximations cited above were developed in the analog domain. With the advent of the digital age, the most straightforward digital implementations were those that "discretized" previously existing analog solutions. Several early papers proposed discretization routines [27–29], and this technique continues to maintain popularity today [30–35]. Building on the success of discretization techniques, parameter optimization is now emerging as a very active area of research [36,37]. While other methods have been introduced (such as the use of the so-called fractional delay operator [38]), this paper focuses its comparisons on discretization and optimization.

Fractional-order digital filters have already been designed, and continue to be designed. However, the literature currently neglects the problem of generating the digital filter coefficients directly in the discrete domain. Direct digital design is desirable, as it is often not possible to simultaneously preserve both the stability and frequency response of analog filters when they are discretized [39]. This paper proposes a method to design fractional-order digital filters directly in the discrete domain through pole-zero placement and subsequent parameter optimization. The poles and zeros are placed logarithmically on the real axis inside the unit circle to maximize performance and guarantee stability.

The remainder of this paper has the following organization. Section 2 provides technical background information, including an overview of selected existing design approaches. Section 3 presents the proposed pole-zero placement algorithm, along with guidelines for parameter selection and optimization. Section 4 examines the results of some example cases, and compares the proposed method with other approaches. Section 5 concludes the paper.

2. Technical background

This section first briefly reviews some fundamental definitions in digital filter design. Next is presented the basic theoretical foundation for fractional-order approximations. Finally, several existing techniques are introduced, both analog and digital.

2.1. Key definitions

This paper follows the common convention of using lower-case f to represent continuous (analog) frequencies and upper-case F to represent normalized discrete (digital) frequencies. The relationship between f and F is given in

$$F = \frac{f}{f_s}, \quad (1)$$

where f_s is the sampling frequency. The Nyquist frequency [40] is denoted by $f_{Nyq} = f_s/2$, and $F_{Nyq} = 0.5$.

An analog filter is represented by its transfer function, as in

$$H(s) = \frac{Y(s)}{X(s)}, \quad (2)$$

where $Y(s)$ and $X(s)$ are the Laplace transforms of the output and input, respectively. In the case of analog lowpass filters, if $Y(s)$ is of degree M, and $X(s)$ is of degree N, the slope of the transition band is $20(M-N)$ dB/decade (dec) or $6(M-N)$ dB/oct [41].

The slope of the magnitude response in the transition band is often referred to as the *rolloff*. Rolloff is often specified in terms of *rolloff order*, which is independent of *filter order*, in the fractional sense. Since M and N are integers by definition, traditional analog lowpass filter design techniques can produce only *integer-order* rolloff (of order $|M-N|$). But applications exist where the desired rolloff is not a multiple of 6 dB/oct, as in IEC standard 60268-16, which governs a speech intelligibility testing method that specifies a 3 dB/oct lowpass filter [2].

2.2. Analog constant-slope approximation

As early as the 1970s, researchers began realizing that careful selection of poles and zeros could approximate a filter with non-integer order rolloff [4]. Alternating poles and zeros in the frequency domain, and varying the relative frequency spacing between them, could produce a somewhat arbitrary slope ranging from zeroth to first order. Furthermore, the quality of the approximation improved with increasing density of pole-zero pairs in the frequency domain [8].

It is well known [41] that any constant slope (analog) transfer function can be decomposed into the product of a rational and irrational function of s, as in

$$s^{-p} = s^{-m} s^{-q}, \quad (3)$$

where m is an integer, p and q are real, and $0 < q < 1$. For typical lowpass transfer functions, m is either zero or positive, indicating m poles at the origin. It is shown in [41] that s^{-q} can be approximated by a rational function with real poles and zeros with alternating logarithmic frequency spacing d_1 and d_2 from zero to pole and pole to zero, respectively. See Fig. 1 for an illustration. This is shown in

$$M = \frac{rise}{run} = \frac{(0d_1 + (-6d_2))\ \text{dB}}{(d_1+d_2)\text{oct}} = \frac{-6d_2}{d_1+d_2}\left(\frac{\text{dB}}{\text{oct}}\right), \quad (4)$$

where M is the desired slope (in dB/oct) approximated by the alternating pole/zero pattern. For any desired slope M,