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A fractional calculus on arbitrary time scales: Fractional differentiation and fractional integration ☆

Nadia Benkhettou [a], Artur M.C. Brito da Cruz [b,c], Delfim F.M. Torres [c,*]

[a] Laboratoire de Mathématiques, Université de Sidi Bel-Abbès B.P. 89, 22000 Sidi Bel-Abbès, Algerie
[b] Escola Superior de Tecnologia de Setúbal Estefanilha, 2910-761 Setúbal, Portugal
[c] Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

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ABSTRACT

We introduce a general notion of fractional (noninteger) derivative for functions defined on arbitrary time scales. The basic tools for the time-scale fractional calculus (fractional differentiation and fractional integration) are then developed. As particular cases, one obtains the usual time-scale Hilger derivative when the order of differentiation is one, and a local approach to fractional calculus when the time scale is chosen to be the set of real numbers.

1. Introduction

Fractional calculus refers to differentiation and integration of an arbitrary (noninteger) order. The theory goes back to mathematicians as Leibniz (1646–1716), Liouville (1809–1882), Riemann (1826–1866), Letnikov (1837–1888), and Grünwald (1838–1920) [24,38]. During the last two decades, fractional calculus has increasingly attracted the attention of researchers of many different fields [1,9,10,29,31,33,35,41].

Several definitions of fractional derivatives/integrals have been defined in the literature, including those of Riemann–Liouville, Grünwald–Letnikov, Hadamard, Riesz, Weyl and Caputo [24,36,38]. In 1996, Kolwankar and Gangal proposed a local fractional derivative operator that applies to highly irregular and nowhere differentiable Weierstrass functions [8,26]. Here we introduce the notion of fractional derivative on an arbitrary time scale $\mathbb{T}$ (cf. Definition 6). In the particular case $\mathbb{T}=\mathbb{R}$, one gets the local Kolwankar–Gangal fractional derivative $\lim_{h\to 0} f(t+h)-f(t)/h^{\alpha}$, which has been considered in [26,27] as the point of departure for fractional calculus. One of the motivations to consider such local fractional derivatives is the possibility to deal with irregular signals, so common in applications of signal processing [27].

A time scale is a model of time. The calculus on time scales was initiated by Aulbach and Hilger in 1988 [7], in order to unify and generalize continuous and discrete analysis [22,23]. It has a tremendous potential for applications and has recently received much attention [3,16,17,20,21]. The idea to join the two subjects — the fractional calculus and the calculus on time scales — and to develop a *Fractional Calculus on Time Scales*, was born with the Ph.D. thesis of Bastos [12]. See also [5,6,13–15,25,37,40] and references therein. Here we introduce a general fractional calculus on time scales and develop some of its basic properties.

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* Corresponding author. Tel.: +351 234370668; fax: +351 234370066.
E-mail addresses: benkhettou_na@yahoo.fr (N. Benkhettou), artur.cruz@estsetubal.ips.pt (A.M.C. Brito da Cruz), delfim@ua.pt (D.F.M. Torres).

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Fractional calculus is of increasing importance in signal processing [35]. This can be explained by several factors, such as the presence of internal noises in the structural definition of the signals. Our fractional derivative depends on the graininess function of the time scale. We trust that this possibility can be very useful in applications of signal processing, providing a concept of coarse-graining in time that can be used to model white noise that occurs in signal processing or to obtain generalized entropies and new practical meanings in signal processing. Indeed, let $\mathbb{T}$ be a time scale (continuous time $\mathbb{T}=\mathbb{R}$, discrete time $\mathbb{T}=h\mathbb{Z}$, $h>0$, or, more generally, any closed subset of the real numbers, like the Cantor set). Our results provide a mathematical framework to deal with functions/signals $f(t)$ in signal processing that are not differentiable in the time scale, that is, signals $f(t)$ for which the equality $\Delta f(t)=f^{\Delta}(t)\Delta t$ does not hold. More precisely, we are able to model signal processes for which $\Delta f(t)=f^{(\alpha)}(t)(\Delta t)^{\alpha}$, $0<\alpha\le 1$.

The time-scale calculus can be used to unify discrete and continuous approaches to signal processing in one unique setting. Interesting in applications, is the possibility to deal with more complex time domains. One extreme case, covered by the theory of time scales and surprisingly relevant also for the process of signals, appears when one fix the time scale to be the Cantor set [11,42]. The application of the local fractional derivative in a time scale different from the classical time scales $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=h\mathbb{Z}$ was proposed by Kolwankar and Gangal themselves: see [27,28] where nondifferentiable signals defined on the Cantor set are considered.

The article is organized as follows. In Section 2 we recall the main concepts and tools necessary in the sequel. Our results are given in Section 3: in Section 3.1 the notion of fractional derivative for functions defined on arbitrary time scales is introduced and the respective fractional differential calculus developed; the notion of fractional integral on time scales, and some of its basic properties, is investigated in Section 3.2. We end with Section 4 of conclusions and future work.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Here we only recall the necessary concepts of the calculus on time scales. The reader interested on the subject is referred to the books [16,17]. For a good survey see [3].

Definition 1. Let $\mathbb{T}$ be a time scale. For $t\in\mathbb{T}$ we define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t):=\inf\{s\in\mathbb{T}: s>t\}$, and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ by $\rho(t):=\sup\{s\in\mathbb{T}: s<t\}$.

Remark 2. In Definition 1, we put $\inf\varnothing=\sup\mathbb{T}$ (i.e., $\sigma(t)=t$) if $\mathbb{T}$ has a maximum t, and $\sup\varnothing=\inf\mathbb{T}$ (i.e., $\rho(t)=t$) if $\mathbb{T}$ has a minimum t, where $\varnothing$ denotes the empty set.

If $\sigma(t)>t$, then we say that t is right-scattered; if $\rho(t)<t$, then t is said to be left-scattered. Points that are simultaneously right-scattered and left-scattered are called isolated. If $t<\sup\mathbb{T}$ and $\sigma(t)=t$, then t is called right-dense; if $t>\inf\mathbb{T}$ and $\rho(t)=t$, then t is called left-dense. The graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t):=\sigma(t)-t$.

We make use of the set $\mathbb{T}^{\kappa}$, which is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^{\kappa}=\mathbb{T}\setminus\{M\}$; otherwise, $\mathbb{T}^{\kappa}=\mathbb{T}$.

Definition 3 (*Delta derivative [2]*). Assume $f:\mathbb{T}\to\mathbb{R}$ and let $t\in\mathbb{T}^{\kappa}$. We define

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(s))-f(t)}{\sigma(s)-t},\quad t\neq\sigma(s),$$

provided the limit exists. We call $f^{\Delta}(t)$ the delta derivative (or Hilger derivative) of f at t. Moreover, we say that f is delta differentiable on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t\in\mathbb{T}^{\kappa}$. The function $f^{\Delta}:\mathbb{T}^{\kappa}\to\mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^{\kappa}$.

Delta derivatives of higher-order are defined in the usual way. Let $r\in\mathbb{N}$, $\mathbb{T}^{\kappa^0}:=\mathbb{T}$, and $\mathbb{T}^{\kappa^i}:=(\mathbb{T}^{\kappa^{i-1}})^{\kappa}$, $i=1,\ldots,r$. For convenience we also put $f^{\Delta^0}=f$ and $f^{\Delta^1}=f^{\Delta}$. The rth-delta derivative f^{Δ^r} is given by $f^{\Delta^r}=(f^{\Delta^{r-1}})^{\Delta}:\mathbb{T}^{\kappa^r}\to\mathbb{R}$ provided $f^{\Delta^{r-1}}$ is delta differentiable.

The following notions will be useful in connection with the fractional integral (Section 3.2).

Definition 4. A function $f:\mathbb{T}\to\mathbb{R}$ is called regulated provided its right-sided limit exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 5. A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$ is denoted by $\mathcal{C}_{rd}$.

3. Main results

We develop the basic tools of any fractional calculus: fractional differentiation (Section 3.1) and fractional integration (Section 3.2). Let $\mathbb{T}$ be a time scale, $t\in\mathbb{T}$, and $\delta>0$. We define the left δ-neighborhood of t as $\mathcal{U}^{-}:=]t-\delta,t[\cap\mathbb{T}$.

3.1. Fractional differentiation

We begin by introducing a new notion: the fractional derivative of order $\alpha\in]0,1]$ for functions defined on arbitrary time scales. For $\alpha=1$ we obtain the usual delta derivative of the time-scale calculus.

Definition 6. Let $f:\mathbb{T}\to\mathbb{R}$, $t\in\mathbb{T}^{\kappa}$, and $\alpha\in]0,1]$. For $\alpha\in]0,1]\cap\{1/q: q \text{ is a odd number}\}$ (resp. $\alpha\in]0,1]\setminus\{1/q: q \text{ is a odd number}\}$) we define $f^{(\alpha)}(t)$ to be the number (provided it exists) with the property that, given any $\epsilon>0$, there is a δ-neighborhood $\mathcal{U}\subset\mathbb{T}$ of t (resp. left δ-neighborhood $\mathcal{U}^{-}\subset\mathbb{T}$ of t), $\delta>0$, such that

$$|[f(\sigma(t))-f(s)]-f^{(\alpha)}(t)[\sigma(t)-s]^{\alpha}|\le\epsilon|\sigma(t)-s|^{\alpha}$$

for all $s\in\mathcal{U}$ (resp. $s\in\mathcal{U}^{-}$). We call $f^{(\alpha)}(t)$ the fractional derivative of f of order α at t.

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