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Duality for the left and right fractional derivatives

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ABSTRACT

We prove duality between the left and right fractional derivatives, independently on the type of fractional operator. Main result asserts that the right derivative of a function is the dual of the left derivative of the dual function or, equivalently, the left derivative of a function is the dual of the right derivative of the dual function. Such duality between left and right fractional operators is useful to obtain results for the left operators from analogous results on the right operators and vice versa. We illustrate the usefulness of our duality theory by proving a fractional integration by parts formula for the right Caputo derivative and by proving a Tonelli-type theorem that ensures the existence of minimizer for fractional variational problems with right fractional operators.

1. Introduction

Differential equations of fractional order appear in many branches of physics, mechanics and signal processing. Roughly speaking, fractional calculus deals with derivatives and integrals of noninteger order. The subject is as old as the calculus itself. In a letter correspondence of 1695, L'Hopital proposed the following problem to Leibniz: "Can the meaning of derivatives with integer order be generalized to noninteger orders?" Since then, several mathematicians studied this question, among them Liouville, Riemann, Weyl and Letnikov. An important issue is that the fractional derivative of order α at a point x is a local property only when α is an integer. For noninteger cases, the fractional derivative at x of a function f is a nonlocal operator, depending on past values of f (left derivatives) or future values of f (right derivatives). In physics, if t denotes the time-variable, the right fractional derivative of $f(t)$ is interpreted as a future state of the process $f(t)$. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development. However, right-derivatives are unavoidable even in physics, as well illustrated by the fractional variational calculus [9,28]. Consider a variational principle, which gives a method for finding signals that minimize or maximize the value of some quantity that depend upon those signals. Two different approaches in formulating differential equations of fractional order are then possible: in the first approach, the ordinary (integer order) derivative in a differential equation is simply replaced by the fractional derivative. In the second approach, one modifies the variational principle by replacing the integer order derivative by a fractional one. Then, minimization of the action leads to the differential equation of the system. This second approach is considered to be, from the standpoint of applications, the more sound one: see, e.g., [7,19]. It turns out that this last approach introduces necessarily right derivatives, even when they are not present in the formulation and data of the problems: see, e.g., [1,18]. Indeed, left and right derivatives are linked by the following integration by parts formula:

$$\int_a^b d^+ f \cdot g \, dt = -\int_a^b f \cdot d^- g \, dt,$$

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over the set of functions f and g admitting right and left derivatives, here generally represented by d^+ and d^- respectively, and such that $f(a)g(a)=f(b)g(b)=0$ (cf. Corollary 4.1 and Remark 4.2). Our duality results provide a very elegant way to deal with such fractional problems, where there exists an interplay between left and right fractional derivatives of the signals.

There are many fields of applications where one can use the fractional calculus [41]. Examples include viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos and fractals (see, e.g., [8,20,24,26,29,35,39,40]). A large (but not exhaustive) bibliography on the use of fractional calculus in linear viscoelasticity may be found in the book [27]. Recently, a lot of attention has been put on the fractional calculus of variations (see, e.g., [3,4,6,14–16,22,23,30–32,37,38]). We also mention [5], where necessary and sufficient conditions of optimality for functionals containing fractional integrals and fractional derivatives are presented. For results on fractional optimal control see, e.g., [2,17,36]. In the present paper we work mainly with the Caputo fractional derivative. For problems of calculus of variations with boundary conditions, Caputo's derivative seems to be more natural because, for a given function y to have continuous Riemann–Liouville fractional derivative on a closed interval $[a,b]$, the function must satisfy the conditions $y(a)=y(b)=0$ [4]. We also mention that if $y(a)=0$, then the left Riemann–Liouville derivative of y of order $\alpha\in(0,1)$ is equal to the left Caputo derivative; if $y(b)=0$, then the right Riemann–Liouville derivative of y of order $\alpha\in(0,1)$ is equal to the right Caputo derivative.

The paper is organized as follows. In Section 2 we present the definitions of fractional calculus needed in the sequel. Section 3 is dedicated to our original results: we introduce a duality theory between the left and right fractional operators. It turns out that the duality between the left and right fractional derivatives or integrals is independent of the type of fractional operator: a right operator applied to a function can always be computed as the dual of the left operator evaluated on the dual function or, equivalently, we can compute the left operator applied to a function as the dual of the right operator evaluated on the dual function. We claim that such duality is very useful, allowing one to directly obtain results for the right operators from analogous results on the left operators, and vice versa. This fact is illustrated in Section 4, where we show the usefulness of our duality theory in the fractional calculus of variations. Due to fractional integration by parts, differential equations containing right derivatives are common in the fractional variational theory even when they are not present in the data of the problems. Here we use our duality argument to obtain a fractional integration by parts formula for the right Caputo derivative (Section 4.1); and we show conditions assuring the existence of minimizers for fractional variational problems with right fractional operators (Section 4.2). We end with Section 5 of conclusions.

Many different dualities are available in the literature. Indeed, duality is an important general theme that has manifestations in almost every area of mathematics, with numerous different meanings. One can say that the only common characteristic of duality, between those different meanings, is that it translates concepts, theorems or mathematical structures into other concepts, theorems or structures, in a one-to-one fashion. For example, [13] introduces the concept of duality between two different approaches to time scales: the delta approach, based on the forward σ operator, and the nabla approach, based on the backward ρ operator [11]. There is, however, no direct connection between the time-scale calculus considered in [13], which is a theory for unification of difference equations (of integer order) with that of differential equations (of integer order), and the fractional (noninteger order) calculus now considered. We are not aware of any published work on the concept of duality as we do here.

2. Brief review on fractional calculus

There are several definitions of fractional derivatives and fractional integrals, like Riemann–Liouville, Caputo, Riesz, Riesz–Caputo, Weyl, Grunwald–Letnikov, Hadamard, and Chen. We present the definitions of the first two of them. Except otherwise stated, proofs of results may be found in [21] (see also [33]).

Let $f:[a,b]\to\mathbb{R}$ be a function, α a positive real number, n the integer satisfying $n-1<\alpha\le n$, and Γ the Euler gamma function. Then, the left Riemann–Liouville fractional integral of order α is defined by

$$ {}_aI_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt $$

and the right Riemann–Liouville fractional integral of order α is defined by

$$ {}_xI_b^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt. $$

The left and right Riemann–Liouville fractional derivatives of order α are defined, respectively, by

$$ {}_aD_x^\alpha f(x)=\frac{d^n}{dx^n}{}_aI_x^{n-\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)\,dt $$

and

$$ {}_xD_b^\alpha f(x)=(-1)^n\frac{d^n}{dx^n}{}_xI_b^{n-\alpha}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (t-x)^{n-\alpha-1}\times f(t)\,dt. $$

The left and right Caputo fractional derivatives of order α are defined, respectively, by

$$ {}_a^CD_x^\alpha f(x)={}_xI_b^{n-\alpha}\frac{d^n}{dx^n}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)\,dt $$

and

$$ {}_x^CD_b^\alpha f(x)=(-1)^n{}_xI_b^{n-\alpha}\frac{d^n}{dx^n}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_x^b (-1)^n(t-x)^{n-\alpha-1}f^{(n)}(t)\,dt. $$