Signal Processing ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

An iterative framework for sparse signal reconstruction algorithms

Q1 Sooraj K. Ambat*, K.V.S. Hari

Statistical Signal Processing Lab, Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560012, India

ARTICLE INFO

Article history:
Received 8 February 2014
Received in revised form
1 August 2014
Accepted 18 September 2014

Keywords:
Compressed sensing
Sparse recovery
Sparse signal
Signal reconstruction
Iterative algorithms

ABSTRACT

It has been shown that iterative re-weighted strategies will often improve the performance of many sparse reconstruction algorithms. However, these strategies are algorithm dependent and cannot be easily extended for an arbitrary sparse reconstruction algorithm. In this paper, we propose a general iterative framework and a novel algorithm which iteratively enhance the performance of any given arbitrary sparse reconstruction algorithm. We theoretically analyze the proposed method using restricted isometry property and derive sufficient conditions for convergence and performance improvement. We also evaluate the performance of the proposed method using numerical experiments with both synthetic and real-world data.

1. Introduction

Compressed Sensing (CS) [1,2] is a new paradigm in signal processing which exploits the sparse or compressible nature of the signal to significantly reduce the number of measurements without compromising on the reconstruction quality. CS uses non-adaptive linear measurements and guarantees robust reconstruction even in the presence of measurement perturbations [3,4]. For this, CS exploits the properties such as sparsity level of the signal and incoherence of the measurement system. Recently many sparse reconstruction algorithms have been proposed in the literature for efficient sparse signal reconstruction. Main works include Convex Relaxation Methods (CRM) [5–7], greedy pursuits [8–11], and Bayesian framework [12–15].

In many applications, partial information about the non-zero locations and the non-zero values of the sparse signal of interest may be available a priori. For example, in signals such as video, the adjacent temporal frames will be highly coherent and a partial knowledge about the support-set of the current frame can be obtained from the estimate of the previously reconstructed frames. In such situations, it has been shown that a better sparsity-measurement trade-off than conventional CRM can be achieved by incorporating this knowledge in the CRM framework [16–19]. This idea has also been extended successfully for other methods to improve the sparsity-measurement trade-off of the existing algorithms [20,21].

The seminal work by Candès et al. [22] showed that, even in the absence of any a priori information, a re-weighted strategy can improve the reconstruction performance of CRM. This method was referred to as Iterative Re-weighted L1 (IRL1). IRL1 exploits the information from the estimated signal in the current iteration to improve the signal reconstruction quality in the subsequent iteration by selectively penalizing the atoms. Many variations of the

* Corresponding author.
E-mail addresses: sooraj@ece.iisc.ernet.in (S.K. Ambat), hari@ece.iisc.ernet.in (K.V.S. Hari).

http://dx.doi.org/10.1016/j.sigpro.2014.09.023
0165-1684/© 2014 Published by Elsevier B.V.

Please cite this article as: S.K. Ambat, K.V.S. Hari, An iterative framework for sparse signal reconstruction algorithms, Signal Processing (2014), http://dx.doi.org/10.1016/j.sigpro.2014.09.023

iterative re-weighted strategies have been proposed recently [23–26]. Unfortunately, none of these iterative strategies cannot be easily extended for an arbitrary Sparse Reconstruction Algorithm (SRA). To the best of our knowledge, there does not exist any general framework for improving the performance of arbitrary SRA, iteratively. In this paper, we propose a general iterative framework to improve the performance of any arbitrary SRA, which we referred to as *Iterative Framework for Sparse Reconstruction Algorithms (IFSRA)*. Similar to IRL1, IFSRA exploits the information from the signal estimate in the current iteration to get a better reconstruction quality in the subsequent iteration.

The organization of the paper is as follows. A brief overview of CS is given in Section 2. In Section 3, we develop the iterative framework and propose IFSRA. We theoretically analyze IFSRA in Section 4 and derive sufficient conditions for improving the signal reconstruction quality. In Section 5, performance of IFSRA is validated using numerical experiments. The notations used in this paper are summarized below.

Notations: Bold upper case and bold lower case Roman letters denote matrices and vectors respectively. Calligraphic letters and Upper case Greek alphabets are used to denote sets. $\|\cdot\|_p$ denotes the pth-norm. $\mathbf{A}_{\mathcal{T}}$ denotes the column sub-matrix of $\mathbf{A}$ formed by the columns of $\mathbf{A}$ listed in the set $\mathcal{T}$. $\mathbf{x}_{\mathcal{T}}$ denotes the sub-vector formed by the elements of $\mathbf{x}$ whose indices are listed in the set $\mathcal{T}$. $(\mathbf{x}_{\mathcal{T}_1})_{\mathcal{T}_2}$ denotes the sub-vector formed by the elements of $\mathbf{x}$ whose indices are listed in the set $\mathcal{T}_1 \cap \mathcal{T}_2$. The best K-sparse approximation to $\mathbf{x}$ is denoted by $\mathbf{x}^K$. Ties are broken lexicographically. $\mathtt{supp}(\mathbf{x})$ denotes the set of indices of non-zero elements in any vector $\mathbf{x}$ and $\mathtt{supp}(\mathbf{x}^K)$ denotes the set of indices of the K largest entries in $\mathbf{x}$. For any two sets $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathcal{T}_1 \setminus \mathcal{T}_2 := \mathcal{T}_1 \cap \mathcal{T}_2^c$ denotes the set difference. $\mathcal{T}^c$ denotes the complement of the set $\mathcal{T}$ w.r.t. the set $\{1, 2, \ldots, N\}$. For a set $\mathcal{T}$, $|\mathcal{T}|$ denotes its cardinality (size), and for a scalar c, $|c|$ denotes the magnitude of c. $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ respectively denote the transpose and pseudo-inverse of matrix $\mathbf{A}$.

2. Background

Consider the standard CS measurement setup where a K-sparse signal $\mathbf{x} \in \mathbb{R}^{N\times 1}$ is measured via $M(\ll N)$ linear measurements

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{w}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{M\times N}$ represents the measurement matrix, $\mathbf{b} \in \mathbb{R}^{M\times 1}$ represents the measurement vector, and $\mathbf{w} \in \mathbb{R}^{M\times 1}$ denotes the additive measurement noise. Though (1) is an underdetermined system, CS theory showed that stable and robust reconstruction of $\mathbf{x}$ is possible if $\mathbf{x}$ is sufficiently sparse and $\mathbf{A}$ satisfies some *incoherence* conditions [1,2]. For example, we can solve the following convex optimization problem to get an estimate of $\mathbf{x}$:

$$\min_{\mathbf{x}} \gamma \|\mathbf{x}\|_1 + \tfrac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2, \tag{2}$$

where $\gamma > 0$ is a pre-fixed regularization parameter. The optimization problem in (2) is widely known as Basis Pursuit Denoising (BPDN) [7] which provides good numerical results and elegant theoretical guarantees. In BPDN, the ℓ_1-term promotes sparsity in the solution whereas the ℓ_2-term ensures consistency in the solution.

In many applications, some *partial knowledge* about the signal may be available a priori. It has been shown that a *weighted version* of (2) often promotes sparsity better in the solution and improves the reconstruction performance in such cases [16–19,27,28]. The weighted ℓ_1-norm minimization form of (2) can be written as

$$\min_{\mathbf{x}} \sum_{i=1}^{N} u_i |x_i| + \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2, \tag{3}$$

where $u_i \geq 0$ denotes the weight at index i. The partial knowledge about the signal can be used for setting different weights, which in turn selectively penalizes different coefficients of the signal.

Even in the absence of such prior information, it has been shown that an iterative re-weighting strategy can result in a better sparsity-measurement trade-off than BPDN. Iterative Re-weighted L1 (IRL1) [22] is one of the early proposed methods in this direction which received wide attention. In the first iteration, IRL1 sets all weights to unity and solves (3). In other words, in the first iteration IRL1 solves (2) (BPDN). Let $\hat{\mathbf{x}}_k$ denote the sparse signal estimated by IRL1 in the kth iteration. In the $(k+1)$th iteration, IRL1 solves (3) with $u_i = 1/(\hat{x}_i + \eta)$ where $\eta > 0$ is a pre-fixed parameter. The iteration continues till some halting condition is reached. Though IRL1 shows significant performance improvement over BPDN, in each iteration IRL1 needs to solve a weighted BPDN and hence IRL1 is computationally much more demanding as compared to BPDN. Many variations of IRL1 have been proposed in the literature to improve the performance and reduce the computational cost. For example, Iterative Support Detection (ISD) [23] uses only binary values (0 or 1) as weights. In each iteration, ISD estimates the indices of the dominant part of the signal known as *active-set* using *thresholding* or by a more sophisticated *first significant jump rule*. The atoms in the active-set are given weights equal to zero and weights of the remaining atoms are set to unity to solve a weighted BPDN in the subsequent iteration. ISD showed a better performance than IRWL1 in both computation time and reconstruction quality.

This idea of exploiting the partial knowledge about the signal to improve the sparse reconstruction has been also extended to other types of sparse reconstruction algorithms to improve the sparsity-measurement trade-off [20,21]. However, to the best of our knowledge, iterative strategies similar to IRL1 are not available for an arbitrary SRA. Next, we develop a general framework which can be used to iteratively improve the sparse reconstruction quality of any SRA.

3. Iterative framework for sparse signal reconstruction

Solving (1) may be viewed as three different tasks related to the elements of $\mathbf{x}$: (i) estimating the sparsity level, (ii) identifying the indices of the non-zero elements, and (iii) estimating the non-zero values. In this paper, we

Please cite this article as: S.K. Ambat, K.V.S. Hari, An iterative framework for sparse signal reconstruction algorithms, Signal Processing (2014), http://dx.doi.org/10.1016/j.sigpro.2014.09.023