Signal Processing ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Improving the time domain response of fractional order digital differentiators by windowing

Chengyan Peng *, Xiaochuan Ma, Geping Lin, Min Wang

State Key Laboratory of Acoustics, Institute of Acoustics, Chinese Academy of Sciences, Beijing 100190, China

ARTICLE INFO

Article history:
Received 8 January 2014
Received in revised form
21 March 2014
Accepted 23 March 2014

Keywords:
Fractional order differentiator
Non-bandlimited signals
Spectrum leakage
Window

ABSTRACT

In this paper, the limitation of using $(j\omega)^{\nu}$ as the ideal frequency response of fractional order digital differentiators for non-bandlimited signals is discussed. High frequency error enhanced by $(j\omega)^{\nu}$, along with the cause of time domain response degradation, is presented. Windows are proposed, which can help to improve the time domain response of fractional order digital differentiator and greatly reduce the filter order if the differentiator is approximated by finite impulse response (FIR) filter. Simulation results show that windowing the output of fractional order digital differentiator in frequency domain is effective in improving the time domain response of signals.

1. Introduction

Fractional order differentiation, which generalizes the order of traditional derivation from integers to real and complex values, has attracted the attentions of researchers in physics and applied sciences in recent years. Engineering applications of fractional order differentiation may be found in electromagnetism [1,2], automatic control [3], circuits and systems [7], as well as digital signal processing [4].

Compared with its integer counterpart, fractional order differentiation offers more flexibility and better design performance in both time domain and frequency domain due to the introduction of fractional order. However, the design procedure of fractional order differentiator is complicated by the complex background of fractional calculus. The state-of-the-art literature implements fractional order operation in continuous and discrete time domains. For continuous time case, the irrational transfer function s^{ν} of fractional order differentiator is approximated by a rational function

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

using evaluation, interpolation and curve fitting techniques. While for discrete time case, the z-domain transfer function is obtained by first replacing s with its discrete equivalent $\omega(z^{-1})$ using Euler method, Al-Alaoui method and Tustin method, etc., and then by expanding $(\omega(z^{-1}))^{\nu}$ using power series method and continued fraction method. Detailed information on approximation of fractional order operator can be found in [9–11].

Basically, there are two methods for realizations of fractional order differentiator. One is digital realization based on microprocessors and the other is analogue realization using special circuits called the fractance. Fractional order differentiator realized in microprocessors, which may be called the fractional order digital differentiator, can be more easily designed provided that the processor meets memory and speed requirements. However, when non-bandlimited input signals are concerned, fractional order digital differentiator suffers significant frequency error, which is seldom discussed in the literature. According to uncertainty principle, a time limited

* Corresponding author.
E-mail address: buptpchy@163.com (C. Peng).

http://dx.doi.org/10.1016/j.sigpro.2014.03.034
0165-1684/

Please cite this article as: C. Peng, et al., Improving the time domain response of fractional order digital differentiators by windowing, Signal Processing (2014), http://dx.doi.org/10.1016/j.sigpro.2014.03.034

signal has infinite frequency bandwidth. Besides, many types of signals are both time unlimited and band unlimited. To reconstruct a non-bandlimited signal from discrete samples, signal spectrum is truncated with frequency components above half of the Nyquist frequency removed, which introduces error to the interpolated time domain signal. Usually, such reconstruction error can be neglected because the discarded frequency components contain insignificant energy. This is not the case when discrete signals are differentiated. As stated in [6,8,12], the ideal frequency response of a digital differentiator is given by

$$H(\omega) = (j\omega)^{\nu}, \quad \omega \in \left(-\frac{\pi}{T_s}, \frac{\pi}{T_s}\right) \tag{1}$$

where T_s is the sampling interval. Eq. (1) shows that the amplitude response of a fractional order digital differentiator has an increasing profile. As a result, the high frequency error of non-bandlimited input signals is enhanced after differentiation. Effect of high frequency error enhancement may be serious when ν is large or input signals contain considerable high frequency energy. To solve this problem, the frequency response of fractional order digital differentiator can be modified, which is part of the discussion in this paper.

On the other hand, the discrete time implementation of fractional order differentiator is in the form of either finite impulse response (FIR) or infinite impulse response (IIR). The IIR implementation is preferred because

- The IIR form contains zeros and poles, which can better approximate the frequency response of fractional order differentiator.
- The order of FIR filter must be very high for good approximation results, which makes the FIR form less efficient in applications. As noted in [15], the filter order should be assigned to 50.

In addition, the time domain response of fractional order differentiator approximated by FIR filter suffers great oscillation, which can be seen from the simulation results in [12] and numerical results in this paper. However, it is well known that FIR filter has no bound on the maximal sampling rate [14] and is always automatically stable because of nonrecursive structures [13]. Thus, if the order of the FIR filter can be reduced and time domain response can be improved, fractional order differentiator in FIR form may achieve as good performance as IIR form does. Finding better FIR filter approximation of fractional order differentiator is the other part of the discussion in this paper.

Following this introduction, this paper first gives some fundamentals of fractional derivatives in Section 2. Then it focuses on the discussion of frequency error associated with fractional order digital differentiators and the selection of windows in Section 3. Simulation results are given in Section 4. Section 5 concludes the paper with some additional remarks.

2. Basics of fractional derivatives

Let us start with some basic concepts commonly used in fractional order digital differentiators. The Grünwald–Letnikov definition of fractional order differentiation for a well-behaved function $f(t)$ is given by [5]

$${}_aD_t^{\nu} f(t) = \lim_{h \to 0} h^{-\nu} \sum_{k=0}^{[(t-a)/h]} (-1)^k \binom{\nu}{k} f(t-kh) \tag{2}$$

where $[\cdot]$ represents the integer part, $\nu > 0$ is the differentiation order and $\binom{\nu}{k} = \nu(\nu-1)\cdots(\nu-k+1)/k!$.

Fractional derivative operator ${}_aD_t^{\nu}$ is often written as D^{ν} when the differentiation limits a and t can be inferred from context. Considering that D^{ν} is linear and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

where $F(\omega)$ is the Fourier transform of $f(t)$, the inverse Fourier transform of $D^{\nu} f(t)$ can be derived as

$$\begin{aligned} D^{\nu} f(t) &= D^{\nu}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) D^{\nu}\left[e^{j\omega t}\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)^{\nu} F(\omega) e^{j\omega t}\, d\omega \end{aligned} \tag{3}$$

where relation $D^{\nu} e^{at} = a^{\nu} e^{at}$ is used in Eq. (3).

For numerical evaluation, the limit operation in Eq. (2) is neglected if h is small enough. In this situation, $D^{\nu} f(t)$ can be approximated by

$$D^{\nu} f(t) \approx \Delta_h^{\nu} f(t) = h^{-\nu} \sum_{k=0}^{[(t-a)/h]} (-1)^k \binom{\nu}{k} f(t-kh) \tag{4}$$

As [15] reveals, $\Delta_h^{\nu} f(t)$ gives a first-order approximation of $D^{\nu} f(t)$.

3. Problem formulation

3.1. Ideal response for fractional order digital differentiator

If a continuous signal $x(t)$ is sampled with pulse train $\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_s)$, the discrete form of $x(t)$ can be written as $x(n) = x(t)|_{t=nT_s}$. To obtain the digital fractional order differentiation of $x(n)$, a digital differentiator with frequency specification $Q_{\nu}(w)$, where ν is the fractional order, should be designed. Fig. 1(top) illustrates the digitization and differentiation process of $x(t)$. The system output is denoted by $y_1(n)$.

Operation in Fig. 1(top) can be viewed from another perspective, which is shown in Fig. 1(bottom). In this case, $x(t)$ is first νth-order differentiated in continuous domain,

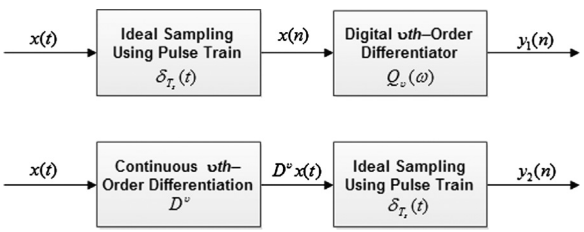

Fig. 1. Procedures for obtaining discrete differentiated signal. Top: signal is first sampled and then digital differentiated. Bottom: signal is first continuously differentiated and then sampled.