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A general framework for sampling and reconstruction in function spaces associated with fractional Fourier transform

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ABSTRACT

The fractional Fourier transform (FRFT) has proven to be a powerful tool in optics and signal processing. Sampling theory of this transform for band-limited signals has blossomed in recent years. However, real-world signals are often not band-limited. In this paper, we first develop the theory of frames for function spaces associated with the FRFT, then we propose a general framework for FRFT-based sampling and reconstruction in function spaces without band-limiting constraints. Based upon the proposed framework, a simple necessary and sufficient condition for FRFT-based uniform sampling in function spaces is found, which facilitates a straightforward derivation of the uniform sampling theorem for the FRFT. The theoretical derivations are validated by the means of numerical results.

1. Introduction

The fractional Fourier transform (FRFT) has proven to be a powerful tool in optics and signal processing. It has many applications [1–9], including in the areas of optics, radar, signal and image processing, communications, etc. As a generalization of the classical Fourier transform (FT), the FRFT can be interpreted as a projection in the time–frequency plane onto a line that forms an angle of α with respect to the time axis [3]. The FRFT of a function $f(t) \in L^2(\mathbb{R})$ is defined as [8]

$$F_\alpha(u) = \mathcal{F}^\alpha\{f(t)\}(u) \triangleq \int_{\mathbb{R}} f(t)\mathcal{K}_\alpha(u,t)\,dt \tag{1}$$

where $\mathcal{F}^\alpha$ denotes the FRFT operator, and kernel $\mathcal{K}_\alpha(u,t)$ is given by

$$\mathcal{K}_\alpha(u,t) = \begin{cases} A_\alpha e^{j((u^2+t^2)/2)\cot\alpha - jtu\csc\alpha}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k-1)\pi \end{cases} \tag{2}$$

where $A_\alpha = \sqrt{(1-j\cot\alpha)/2\pi}$ and $k \in \mathbb{Z}$. The u-axis is regarded as the FRFT domain. Many properties of the FRFT are already known [3,7,8]. The main property of this transform lies in the rotation property, i.e., $\mathcal{F}^\alpha\{\mathcal{F}^\beta\{f(t)\}\} = \mathcal{F}^{\alpha+\beta}\{f(t)\}$. Consequently, the FRFT is invertible, whose inverse version with respect to angle α is the FRFT with angle $-\alpha$, i.e.,

$$f(t) = \mathcal{F}^{-\alpha}\{\mathcal{F}^\alpha\{f(t)\}\}(t) = \int_{\mathbb{R}} F_\alpha(u)\mathcal{K}_\alpha^*(u,t)\,du \tag{3}$$

where $*$ in the superscript indicates the complex conjugate. In general, we only consider the case of $0<\alpha<\pi$, since (1) can easily be extended outside the interval $[0,\pi]$ by noting that $\mathcal{F}^{2\pi n}$ is the identity operator for any $n \in \mathbb{Z}$

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and the rotation property. When $\alpha=\pi/2$, (1) reduces to the classical FT.

Sampling refers to the process of reconstructing a continuous-time signal from its samples. The best known result is the Shannon sampling theorem [11–13] for band-limited signals based upon the FT. Since the FRFT is a generalization of the FT, various sampling expansions for signals band-limited in the FT domain have been extended to the FRFT domain [14–25]. However, many analog signals encountered in practical engineering applications are non-bandlimited. Towards this end, Bhandari and Marziliano [26] proposed a sampling and reconstruction scheme for sparse signals which are non-bandlimited in the FRFT domain, and specifically modeled as a continuous-time periodic stream of Diracs. In [27], Liu et al. derived new sampling expansions for non-bandlimited signals by introducing certain types of non-bandlimited function spaces associated with the linear canonical transform, which is a generalized form of the FRFT. Unfortunately, as the authors of [27] pointed out, there are no normative rules for determining the parameters of the non-bandlimited function spaces in practical implementations at present. In [28], Bhandari and Zayed constructed a class of sampling spaces associated with the FRFT and also derived conditions on the existence of orthonormal and Riesz bases in the spaces. However, the sampling theorem in the sampling spaces was not addressed. In our recent work [29], we established a sampling theorem for the FRFT without band-limiting constraints. The proposed sampling theorem, however, applies only to the case of Riesz bases. The major contributions of this paper, relative to other related work in the literature and our own previous work, are threefold:

(1) we develop the theory of frames for function spaces associated with the FRFT;
(2) we propose a general framework for FRFT-based sampling and reconstruction in function spaces, which places no restrictions on the input signal; and
(3) a simple necessary and sufficient condition for FRFT-based uniform sampling in function spaces is found, which facilitates a straightforward derivation of the uniform sampling theorem for the FRFT.

The remainder of this paper is organized as follows. In Section 2, notation and some facts of frame theory are given, and the discrete-time FRFT is briefly introduced. In Section 3, a general framework of sampling and reconstruction in function spaces associated with the FRFT is proposed. Then, the proposed framework is applied to FRFT-based uniform sampling in function spaces in Section 4. Numerical results are presented in Section 5. Finally, concluding remarks are drawn in Section 6.

2. Preliminaries

2.1. Notation

Continuous signals are denoted with parentheses, e.g., $f(t)$, $t\in\mathbb{R}$, and discrete signals with brackets, e.g., $q[n]$, $n\in\mathbb{Z}$. The scalar product of two functions $f(t)$ and $g(t)$ in $L^2(\mathbb{R})$ is defined as $\langle f,g\rangle=\int_{\mathbb{R}} f(t)g^*(t)\,dt$, and the norm of a function $f(t)\in L^2(\mathbb{R})$ is defined as $\|f\|=\langle f,f\rangle^{1/2}$. For a measurable function $f(t)$ on $\mathbb{R}$, let $\|f(t)\|_\infty=\text{ess sup}|f(t)|$ and $\|f(t)\|_0=$ ess inf$|f(t)|$ be the essential supremum and infimum of $|f(t)|$, respectively. The characteristic function of a measurable subset $E\subset\mathbb{R}$ is denoted with $\chi_E(t)$, where $\chi_E(t)=1$, $t\in E$, and 0 otherwise.

2.2. Basic facts of frame theory

Let $\mathcal{H}$ be a subspace of $L^2(\mathbb{R})$. A function sequence $\{\varphi_n(t)\}_{n\in\mathbb{Z}}\subset\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exists a constant $C\ge 1$ such that

$$C^{-1}\|f\|^2\le\sum_{n\in\mathbb{Z}}|\langle f(t),\varphi_n(t)\rangle|^2\le C\|f\|^2 \tag{4}$$

holds for any $f(t)\in\mathcal{H}$. A frame is said to be exact if it ceases to be a frame when any one of its elements is removed. It is known [30] that a frame is exact if and only if it is a Riesz basis for $\mathcal{H}$. For any frame $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ of $\mathcal{H}$, there exists a so-called dual frame $\{\tilde{\varphi}_n(t)\}_{n\in\mathbb{Z}}\subset\mathcal{H}$ such that

$$f(t)=\sum_{n\in\mathbb{Z}}\langle f(t),\tilde{\varphi}_n(t)\rangle\varphi_n(t) \tag{5}$$

holds in $L^2(\mathbb{R})$ for any $f(t)\in\mathcal{H}$. Taking a linear operator T on $\mathcal{H}$ leads to

$$T\{f(t)\}=\sum_{n\in\mathbb{Z}}\langle f(t),\varphi_n(t)\rangle\varphi_n(t). \tag{6}$$

Then, $\langle T\{f(t)\},f(t)\rangle=\sum_{n\in\mathbb{Z}}|\langle f(t),\varphi_n(t)\rangle|^2$. Eq. (4) implies that the operator T is bounded, self-conjugate, and invertible. It is easy to see that the function sequence $T^{-1}\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ is a dual frame of frame $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$, and T is called a frame transform of $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$. The scalar sequence $\{\langle f(t),\varphi_n(t)\rangle\}_{n\in\mathbb{Z}}$ is called a moment sequence of $f(t)$ to frame $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$. Let $f(t)=\sum_{n\in\mathbb{Z}}c_n\varphi_n(t)$. If $\{c_n\}_{n\in\mathbb{Z}}$ is a moment sequence of a function to $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$, then it must be

$$c_n=\langle T^{-1}\{f(t)\},\varphi_n(t)\rangle,\quad \forall n\in\mathbb{Z} \tag{7}$$

due to the fact that $c_n=\langle h,\varphi_n(t)\rangle$ for some function $h(t)\in\mathcal{H}$, and in $L^2(\mathbb{R})$

$$T^{-1}\{f(t)\}=\sum_{n\in\mathbb{Z}}\langle h(t),\varphi_n(t)\rangle T^{-1}\{\varphi_n(t)\}=h(t). \tag{8}$$

2.3. The discrete-time FRFT

There are two different definitions [10,16] for the discrete-time FRFT (DTFRFT) in the literature. We adopt the one introduced in [16], which has a simple structure. The DTFRFT of a sequence $\{q[n]\}_{n\in\mathbb{Z}}$ is defined as [16]

$$\tilde{Q}_\alpha(u)=\sum_{n\in\mathbb{Z}}q[n]\mathcal{K}_\alpha(u,n). \tag{9}$$

It is clear that if $\{q[n]\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$, $\tilde{Q}_\alpha(u)\in L^2[0,2\pi\sin\alpha]$. Conversely, the inverse DTFRFT is given by $q[n]=\int_I\tilde{Q}_\alpha(u)\mathcal{K}^*_\alpha(u,n)\,du$, where $I\triangleq[0,2\pi\sin\alpha]$. The DTFRFT has the following chirp-periodicity [29]:

$$\tilde{Q}_\alpha(u+2k\pi\sin\alpha)e^{-j((u+2k\pi\sin\alpha)^2/2)\cot\alpha}=\tilde{Q}_\alpha(u)e^{-j(u^2/2)\cot\alpha},\quad \forall k\in\mathbb{Z}. \tag{10}$$

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