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Fractional Alexander polynomials for image denoising

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ABSTRACT

Image denoising is an important task in image processing. The interest in using a fractional mask window operator based on fractional calculus has grown for image denoising. This paper mainly introduces the concept of fractional calculus and proposes a new mathematical method in using fractional Alexander polynomials for image denoising. The structures of $n \times n$ fractional mask windows on eight directions of this algorithm are constructed. Finally, we measure the denoising performance by employing experiments based on visual perception and by using peak signal-to-noise ratios. The experiments illustrate that the improvements achieved are compatible with other standard smoothing filters.

1. Introduction

Noise is any undesired signal that contaminates an image. Digital image acquisition is the primary process by which noise appears in digital images, converting an optical image into a continuous electrical signal. Noise, arising from a variety of sources, is inherent to all electronic image sensors and electronic components in the image environment. The goal of image denoising methods is to recover the original image contaminated by noise. Removing noise from the original signal remains an interesting topic for researchers. Several methods have been proposed to remove the noise and recover the true image, with each approach having its advantages and limitations [1].

Image denoising refers to the process of recovering a digital image that has been contaminated by all kinds of noise, while preserving as much as possible the textures and edges present in the image. Image denoising considered as an important task in image segmentation, feature extraction, and texture analysis. Traditionally, linear models, such as the Gaussian filter, have been commonly used to reduce noise. These methods perform well in the flat regions of images. However, their limitation is the inability to well-preserve the edges. The nonlinear model, however, can handle edges better than linear models. Another denoising method known as neighborhood filtering preserves a pixel by obtaining the average of the values of its neighbors [2]. Recently, many scholars have applied the theory of fractional calculus to image processing. Fractional calculus and its applications are important in several diverse areas of mathematical, physical, and engineering sciences. Fractional calculus generalizes ideas of the calculus of integrals and the derivatives of any arbitrary real or complex order. The advantages of fractional derivatives are obvious in modeling the mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids, rocks, and in many other fields [3,4].

Studies on fractional calculus that involve different operators, such as Riemann–Liouville, Erd'elyi–Kober, Weyl–Riesz, Caputo, and Grünwald–Letnikov operators, have evolved during the past 40 years, and have extended in other fields. Fractional calculus in the field of image processing has gotten considerable attention in image

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texture enhancement [5–7] and image denoising [8–11]. All the results that are based on fractional calculus operators showed that these methods are effective and reliable, and resulted in high levels of permanent immunity against different types of noise.

The fractional integral is extensively used in image denoising algorithms. Hu et al. [8,12] proposed a fractional integral denoising algorithm and the implementation of a fractional integral filter using fractional integral mask windows on eight directions based on fractional calculus Riemann–Liouville definition. Simulation experiments showed the feasibility of the proposed fractional integral denoising algorithm. Guo et al. [13] proposed an image denoising algorithm using fractional integral mask windows based on the Grünwald–Letnikov definition of fractional calculus. Grünwald and Letnikov achieved fine-tuning of image denoising by setting a smaller fractional order and controlled the effect of image denoising by iteration.

In our previous study [9], we proposed a novel digital image denoising algorithm based on the generalized Srivastava–Owa fractional integral operator. The results illustrated that the proposed algorithm has a good upgrading of the denoised image. For image texture enhancement, Jalab and Ibrahim [5] proposed a texture enhancement technique using the fractional order Savitzky–Golay differentiator. This technique computes the generalized fractional order derivative of the input image using the sliding weight window over the image. Jalab and Ibrahim [6] proposed a texture enhancement technique for medical images using fractional differential mask windows based on the Srivastava–Owa fractional operators. Pu et al. [14] proposed fractional differential mask windows based on Grünwald–Letnikov and Riemann–Liouville for multiscale texture enhancement using six fractional differential masks. Experiments proved that the nonlinearly enhancing complex texture in a smooth area by fractional differential-based approach approves to be visibly better than that by traditional integral-based algorithms. Gao et al. [15] proposed image enhancement based on improved fractional differentiation by piecewise quadratic interpolation equation. Experiments showed that for texture-rich digital image, the capability of nonlinearly enhancing comprehensive texture details by improved fractional differentiation is obvious.

In this paper, we utilize the concept of fractional calculus for image denoising to generalize the Alexander polynomial in two approaches, namely, Alexander polynomial–fractional differential (AFD) and Alexander polynomial–fractional integral (AFI).

The Alexander polynomials advantage over the other techniques is that these polynomials can be computed by utilizing a skein relation, which can be employed in various topics in mathematics and physics, such as operator algebras and statistical mechanics [16].

The structures of $n\times n$ fractional mask windows of these algorithms are constructed. The denoising performance is measured by employing experiments based on the visual perception and by using peak signal-to-noise ratio (PSNR). The remainder of this paper is organized as follows. In Section 2, we introduce the generalized fractional differential and fractional integral of the Alexander polynomial. The construction of the fractional differential mask windows, which is the new method proposed in this work, is presented in Section 3. The experimental results and the comparison with other studies are shown in Sections 4 and 5, respectively. Finally, the conclusion is presented in Section 6.

2. Alexander polynomial

The Alexander polynomial is a knot invariant created in 1923 by J.W. Alexander, with integer coefficients corresponding to each knot type. The Alexander polynomial was the only known knot polynomial until the Jones polynomial was derived in 1984. The Alexander polynomial is the main tool used to discuss a pair of curves known as a Zariski pair. This pair can be defined as follows: a couple of curves C_1 and C_2 of equal degree is used to design a Zariski pair. If neighborhoods exist, then $T(C_i)\subset P^2$ (projective plane) of C_i, $i=1,2$ such that $(T(C_1,\ C_1))$ and $(T(C_2,C_2))$ are diffeomorphic, while the pairs (P^2,C_1) and (P^2,C_2) are not homeomorphic (topologically not equivalent). Our aim is to construct two types of mask windows utilizing the Alexander polynomial and its generalization.

Definition 1. The Alexander polynomial is written as [17]

$$\Delta(t)=\prod_{n=1}^{d-1}\Delta_n(t)^{\ell_n},\quad n=1,\dots,d-1 \tag{1}$$

where ℓ_n is positive integer and

$$\Delta_n(t)=\left(t-\exp\left(\frac{2n\pi i}{d}\right)\right)\left(t-\exp\left(-\frac{2n\pi i}{d}\right)\right).$$

From (1), we can conclude the following $\Delta_n(t)$:

$$\begin{aligned}
&\Delta_1=\Delta_{11}=t^2-\sqrt{3}t+1,\\
&\Delta_2=\Delta_{10}=t^2-t+1,\\
&\Delta_3=\Delta_9=t^2+1,\\
&\Delta_4=\Delta_8=t^2+t+1,\\
&\Delta_5=\Delta_7=t^2+\sqrt{3}t+1,\\
&\Delta_6=(t+1)^2.
\end{aligned} \tag{2}$$

2.1. Fractional calculus

The idea of fractional calculus was proposed over 300 years ago. Abel, in 1823, investigated the generalized tautochrone problem and, for the first time, applied fractional calculus techniques in a physical problem. Liouville subsequently applied fractional calculus to problems in potential theory. Since that time, fractional calculus has captured the attention of many researchers in all areas of sciences [4].

This subsection deals with some preliminaries and notations regarding fractional calculus.

Definition 2. The fractional (arbitrary) order integral of the function f of order $\alpha>0$ is defined by

$$I_a^\alpha f(t)=\int_a^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)d\tau.$$

when $a=0$, we write $I_a^\alpha f(t)=f(t)*\phi_\alpha(t)$, where $(*)$ denoted the convolution product, $\phi_\alpha(t)=(t^{\alpha-1}/\Gamma(\alpha))$, $t>0$ and