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Brief paper

Stability analysis of linear systems under state and rate saturations☆

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ABSTRACT

In this paper, we provide a stability analysis for linear systems which have simultaneous saturations in the states and their rates (dynamics). The stability analysis is conducted via a formulation based on the polytopic representations of two saturation functions, state and rate saturation functions. Sufficient conditions to guarantee local and global asymptotical stability of such systems are derived. Furthermore, using the invariance analysis, we estimate the domain of attraction of the system when it is locally asymptotically stable. The stability conditions are formulated as an iterative LMI (ILMI) algorithm. Finally, numerical examples are presented to validate the proposed method.

1. Introduction

The dynamical system with saturation nonlinearities has received a great amount of research interest due to its frequently occurrence in various engineering and scientific problems. For examples, control systems with saturating actuators and sensors (Berstein & Michel, 1995; Cao, Lin, & Chen, 2003; Hu & Lin, 2001), digital filters with saturation overflow arithmetic (Liu & Michel, 1992), and neural networks defined on hypercubes (Jin, Nikiforuk, & Gupta, 1994; Michel, Si, & Yen, 1991) can be modeled with saturation nonlinearities. Specifically, the actuator saturation problems have been popularly and steadily studied by many researchers during the last two decades (Alamo, Cepeda, & Limon, 2005; Bateman & Lin, 2003; Berstein & Michel, 1995; Cao, Lam, & Sun, 1998; Hindi & Boyd, 1998; Hu & Lin, 2001; Pittet, Tarbouriech, & Burgat, 1997; Tarbouriech, Prieur, & Gemes da Silva, 2006; Zhou, Zheng, & Duan, 2011). It is well known that actuator saturation affects not only a system performance, but also stability of the system; and hence, it has been one of the important issues in control engineering to handle actuator saturations. Mainly, there have been two key approaches to deal with actuator saturation problems. The first approach can be called *sector-based analysis*. Based on the sector-boundedness of saturation nonlinearity, the stability conditions and domain of attraction have been studied (Hindi & Boyd, 1998; Pittet et al., 1997; Tarbouriech et al., 2006). The other one is called *linear differential inclusion (LDI) approach*. It was developed on the basis of the polytopic representation of saturation nonlinearity by exploring a special property of the saturation function (Hu & Lin, 2001), and was extended to nested saturation problems (Bateman & Lin, 2003; Mesquine, Taeo, & Benzaouia, 2004; Zhou et al., 2011).

In the territory of saturation nonlinearities, in certain physical systems, the phenomenon of the limitation in states frequently appears. The limitation in states takes place in order to protect the system within a stable area or to restrict the system's operation range to physical constraints of the devices. Therefore, such physical systems can be modeled with state saturation defined on a closed hypercube $\mathbf{D}^n$. From a literature search, it is seen that the state saturation problems have been studied extensively (Fang & Lin, 2004; Hou & Michel, 1998; Ji, Sun, & Liu, 2008; Liu & Michel, 1994; Mantri, Saberi, & Venkatasubramanian, 1998). In more detail, for the second-order systems with state saturation, necessary and sufficient conditions for global asymptotic stability were established in Hou and Michel (1998) and Mantri et al. (1998). Similar to actuator saturation problem, the polytopic representation of state saturation was investigated in Fang and Lin (2004) and Ji et al. (2008), where a less conservative condition for higher-order systems with state saturation was developed. Similarly to the state saturation, rate (dynamics) saturation problems have been also studied (Albertini & D'Alessandro, 1996; Hu & Lin, 2000). For the rate (dynamics) saturation, a sufficient condition for global asymptotic stability was derived in Albertini and D'Alessandro (1996). As a special case, necessary and sufficient

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condition of the second-order systems for a globally asymptotically stability was investigated in Hu and Lin (2000).

In this paper, we are interested in linear systems that have simultaneous saturations in the states and their rates (dynamics). Such systems can be considered as a generalization of nested saturated systems, since the saturation function includes another additional saturation behavior. However, because the system with simultaneous saturations in the states and their rates (dynamics) can be defined only within the closed hypercube $\mathbf{D}^n$, it should be distinguished from the nested saturated systems. We may find the examples of such systems in real applications. For instance, in describing the dynamics of a car, speed and steer angle can be chosen as the state variables. Since both the variables have upper and lower limits, the system has state saturations (Liu & Michel, 1992). Furthermore, acceleration and angular velocity are limited, too. Thus, the dynamics of a car can be modeled with state and rate saturations. Note that the systems with simultaneous saturations in the states and their rates may be used to represent physical systems including dynamic actuators. Consider the power systems controlled by the speed governor. Realistic governor models may include limits on the valve position and its displacement (Siljak, Stipanovic, & Zecevic, 2002). Therefore, the power systems may be modeled with partial state and rate saturations also.

This paper takes the similar approach to those in Zhou et al. (2011), but extends the results to two different saturation nonlinearities. The basic idea of the paper is to integrate two polytopic differential inclusions. That is, by convexity of each vertex in one inclusion, we can test all the vertices of the other ones. Consequently, the contribution of this paper can be compactly summarized as follows. First, based on the polytopic representations of two different saturation functions, we derive sufficient conditions to guarantee local and global stability. Second, the condition for the invariance of the intersection of two sets $\mathcal{E}(P)$ and $\mathbf{D}^n$ is derived (see Section 2 for the notations), and thus, we estimate the domain of attraction from the intersection of two sets. Third, we provide an algorithm for stability test and for the estimation of the domain of attraction via iterative linear matrix inequalities (ILMIs).

The paper is structured as follows. First, we define the state and rate saturated systems, and provide polytopic representations of state and rate saturation functions in Section 2. Then, we derive local and global asymptotical stability conditions in Section 3. In order to evaluate the asymptotical stability and to estimate the domain of attraction, we provide an ILMI algorithm in Section 4. In Section 5, numerical example is presented, and then conclusion and future works are discussed in Section 6, consequently.

2. Problem statement and preliminaries

In this paper, we use the following notations. The i-th component of a vector x is denoted by $x_{(i)}$. For a matrix $A \in \mathbf{R}^{m\times n}$, A^T denotes the transpose of A. The elements of the matrix A are denoted by $A_{(i,j)}$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, and $A_{(i,:)}$ and $A_{(:,i)}$ denote the i-th row and i-th column of matrix A, respectively. For a symmetric positive-definite matrix $P \in \mathbf{R}^{n\times n}$, we define an ellipsoid as $\mathcal{E}(P) = \{x \in \mathbf{R}^n : x^TPx \le 1\}$. Let $H \in \mathbf{R}^{n\times n}$, we then define $\mathcal{L}(H) = \{x \in \mathbf{R}^n : |H_{(i,:)}x| \le 1,\ i = 1, 2, \ldots, n\}$ which implies the linear region of the saturation function sat(Hx). The symbol co$\{\cdot\}$ denotes the convex hull. We denote a hypercube by $\mathbf{D}^n = \{x \in \mathbf{R}^n : -1 \le x_{(i)} \le 1,\ i = 1, 2, \ldots, n\}$, and its boundary by $\partial\mathbf{D}^n = \bigcup_{i=1}^n \partial\mathbf{D}_i$, where $\partial\mathbf{D}_i = \{x \in \mathbf{R}^n : |x_{(i)}| = 1, -1 \le x_{(j)} \le 1, j = 1, 2, \ldots, i-1, i+1, \ldots, n\}$. For $\partial\mathbf{D}_i$, denote $\partial\mathbf{D}_i^+ = \{x \in \mathbf{R}^n : x_{(i)} = 1, -1 \le x_{(j)} \le 1, j = 1, 2, \ldots, i-1, i+1, \ldots, n\}$, $\partial\mathbf{D}_i^- = \{x \in \mathbf{R}^n : x_{(i)} = -1, -1 \le x_{(j)} \le 1, j = 1, 2, \ldots, i-1, i+1, \ldots, n\}$, and their vertices as vert$(\partial\mathbf{D}_i^+)$ and vert$(\partial\mathbf{D}_i^-)$, respectively.

Similarly to Zhou et al. (2011), we use some special notations. For two integers $p \ge 1$ and $n \ge 1$, we define the symbol $\mathcal{V}_p^n$ as a set such as $\mathcal{V}_p^n = \{v = [v_{(1)}, v_{(2)}, \ldots, v_{(n)}]^T \in \mathbf{R}^n : v_{(i)} \in \{1, 2, \ldots, p\}, i = 1, 2, \ldots, n\}$, which contains p^n elements. For example, if $p = 3$, $n = 2$, then

$$\mathcal{V}_3^2 = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix} \right\}.$$

Furthermore, let $\mathcal{D}^n$ be the set of $n \times n$ diagonal matrices whose diagonal elements are either 1 or 0. We define $D_l^v \in \mathcal{D}^n, l = 1, 2, \ldots, p$, such that if $v_{(i)} = l$, then the i-th diagonal elements of matrix D_l^v are 1 and the others are zeros. For example, for given $\mathcal{V}_3^2$, if $v = [3, 2]^T \in \mathcal{V}_3^2$, then

$$D_1^v = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \qquad D_2^v = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \qquad D_3^v = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

In this paper, we consider the following linear systems that have simultaneous saturations in the states and their rates (dynamics):

$$\dot{x} = \text{sat}\,(h(Ax))\,, \tag{1}$$

where $x \in \mathbf{D}^n, A \in \mathbf{R}^{n\times n}$ is a real matrix, and two saturation functions, $h(\bullet)$ and sat$(\bullet)$, represent state and rate saturations defined as follows:

$$h(Ax) = \begin{bmatrix} h\left(\sum_{j=1}^n A_{(1,j)}x_{(j)}\right) \\ h\left(\sum_{j=1}^n A_{(2,j)}x_{(j)}\right) \\ \vdots \\ h\left(\sum_{j=1}^n A_{(n,j)}x_{(j)}\right) \end{bmatrix}, \tag{2}$$

with, for each $i = 1, 2, \ldots, n$, the following symmetric constraints

$$h\left(\sum_{j=1}^n A_{(i,j)}x_{(j)}\right) = \begin{cases} 0, & \text{if } |x_{(i)}| = 1 \text{ and } \left(\sum_{j=1}^n A_{(i,j)}x_{(j)}\right) x_{(i)} > 0 \\ \sum_{j=1}^n A_{(i,j)}x_{(j)}, & \text{otherwise} \end{cases} \tag{3}$$

and

$$\text{sat}(\phi_{(i)}) = \text{sign}(\phi_{(i)}) \min(1, |\phi_{(i)}|), \tag{4}$$

respectively.

It is well known that for rate (dynamics) saturated systems (i.e., $\dot{x} = \text{sat}(Ax)$) and state saturated systems (i.e., $\dot{x} = h(Ax)$), Hurwitz on A does not imply a global stability within $\mathbf{D}^n$ (Albertini & D'Alessandro, 1996; Hu & Lin, 2000; Mantri et al., 1998). The objective of this paper is, thus, to study local and global stability of the system (1) as well as to estimate the domain of attraction of the origin. In what follows, we first provide some definitions and lemmas that are useful for our main results. For the saturated systems, invariance method (Blanchini, 1999) is widely used for stability analysis, synthesis, and for the estimation of the domain of attraction.

Consider the following time-invariant systems

$$\dot{x} = f(x), \tag{5}$$