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Undamped oscillations in fractional-order Duffing oscillator

Mohammad Rostami [a,*], Mohammad Haeri [b]

[a] Department of Electrical and Systems Engineering, University of Pennsylvania, Philadelphia, USA
[b] Electrical Engineering Department, Sharif University of Technology, Tehran, Iran

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ABSTRACT

This paper studies undamped oscillations of fractional-order Duffing system. Stability theorems for fractional order systems are used to determine the characteristic polynomial of the system in order to find the parametric ranges for undamped oscillations in this system. We also derive relations for estimating the frequency and the amplitude of the oscillations in this system using a describing function method. Finally numerical simulation results are provided to justify the analysis.

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1. Introduction

Definition of derivative with respect to an integer order can be extended to real order derivatives (as well as complex orders) in a well-defined manner. Consequently fractional order concepts, e.g. fractional order differential equations, can be defined along with integer order concepts. Fractional order calculus has been the subject of pure mathematics research for long time but only in the past three decades its applications in modeling the real world phenomenon as well as engineering emerged [1]. Integer order differentiation is a local operator, that is, its value at a given point depends only on values of function near that point. In contrast, fractional differentiation is not a local operator in general and thus potentially can outperform integer order differentiation when we deal with properties like memory. It has been shown that elements with memory in the nature, e.g. capacitors, can be modeled better using fractional order differential equations [2]. Fractional order calculus has been applied to different applications in signal processing to improve classical signal processing algorithms which use integer order calculus, including edge detection [3], image segmentation [4], wavelet theory [5], filter design [6], and image denoising [7].

Studying dynamical behavior of fractional systems has attracted considerable attention in recent years. For instance, dynamics of population densities are generally described using exponential laws, whereas there are systems that dynamics obeys faster or slower-than-exponential laws. In such cases the dynamics may be described better by fractional order systems [8]. Viscoelasticity is another area of applications of fractional dynamics where the material exhibits its nature between purely elastic and pure fluid [9]. Anomalous diffusion is another example where fractional-order diffusion equation can describe the anomalous flow in the diffusion process [10]. Many fractional order systems are inspired from extension of integer order systems [11–17]. In such cases, we are interested to study the effect of the fractional derivative on behavior of the system and verify the differences. Study of different cases indicates fractional version of an integer order system may have different behavior compared to the original integer order system. For instance it has been shown that a fractional order Brusselator oscillator with an order less than two can have oscillatory behavior [11], whereas in ordinary integer systems the minimum order must be two. It has been shown that in the fractional Van der Pol system, trajectory of

* Corresponding author. Tel.: +1 215 746 5425; fax: +1 215 898 7301.
E-mail address: mrostami@seas.upenn.edu (M. Rostami).

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the oscillations depends on the initial value of the system [12,13]. It has been shown that a fractional-order Wien-bridge oscillator can generate a limit cycle for any fractional order [14] subject to proper selection of the amplifier gain. It has been shown that the fractional-order Chua system with order as low as 2.7 produces chaotic oscillations [15]. There is an ongoing research to study and analyze this class of fractional systems.

In the current work, the fractional order Duffing system is studied [18]. Oscillatory and chaotic behavior of this system has been studied extensively in the literature [19–22]. The model is simple but can have regular and chaotic oscillations and is a common second order systems, studied in the literature. The Duffing system has been extended to a fractional system in the literature [23–29]. Several scenarios can be used to fractionalize the Duffing system. In [23,24], authors modify the Duffing system by coupling two normal Duffing systems and fractionalize the modified Duffing system in a state space. Using numerical simulations, Ref. [23] shows that a fractional modified Duffing system indicates chaotic behavior. In [24], authors extend [23] by considering a periodically forced case and use numerical simulations to detect chaotic behavior without further mathematical analysis. Another group of authors fractionalize the Duffing system by considering a fractional damping term [25–29]. It has been shown that this extension could indicate chaotic and regular oscillations [25–27]. In [28,29], authors study resonance in this system. In the current work, we extend the generalization of [25–29] by further fractionalizing the Duffing system in state space domain. We focus on analyzing the regular oscillatory behavior of this system using mathematical tools. Based on stability theorems derived for fractional order systems, a parametric region for undamped oscillations is derived analytically. We also derive relations for estimating the frequency and the amplitude of the oscillations in this system using a describing function method. This approach has been used in the literature to study the fractional order Van der Pol oscillator [12,13], and the fractional order Arneodo oscillator [16]. The analysis is further investigated via numerical simulations.

The paper is organized as follows. In Section 2 some preliminary mathematical concepts are reviewed. In Sections 3 the fractional order Duffing system is introduced and the parametric region for undamped oscillation is derived. Section 4 is devoted to estimating the frequency and the amplitude of the oscillations using the describing function method. Numerical simulation results are presented in Section 5. The paper is finally concluded in Section 6.

2. Basic preliminaries

Fractional derivative is defined as the extension of integer order derivative. The extension must be a well-defined linear operator. Several definitions are proposed for fractional derivative. Common definitions include Riemann–Liouville, Grunwald–Letnikov, and Caputo definitions [1].

Let $f(\cdot)$ be a real valued integrable function. The ath $\in \mathbb{R}^+$ Riemann–Liouville fractional derivative of the function $f(\cdot)$ with respect to t is defined as

$$_{RL}D_t^a f(t) = \frac{d^n}{dt^n} J^{n-a} f(t), \tag{1}$$

where $n \in \mathbb{N}$ is the integer for which $n-1 \le a < n$ and J is the fractional integral of the function $f(\cdot)$ defined by

$$J^a f(t) = \frac{1}{\Gamma(a)} \int_0^t (t-x)^{a-1} f(x)\, dx, \tag{2}$$

where $\Gamma(a) = \int_0^\infty e^{-t} t^{a-1}\, dt$ is the Gamma function.

The ath Grunwald–Letnikov fractional derivative of the function $f(\cdot)$ with respect to t and the terminal value 0 is defined by

$$_{GL}D_t^a f(t) = \lim_{N\to\infty} \left[\frac{t}{N}\right]^a \sum_{j=0}^{N} (-1)^a \binom{a}{j} f\left(t - j\left[\frac{t}{N}\right]\right), \tag{3}$$

where $\binom{a}{j} = \Gamma(a+1)/j!\Gamma(a+1-j)$.

The Caputo definition of the fractional derivative is defined as

$$_C D_t^a f(t) = \begin{cases} J^{n-a} \frac{d^n}{dt^n} f(t), & n-1 < a < n \\ \frac{d^n}{dt^n} f(t), & a = n \end{cases}, \tag{4}$$

where $a \in \mathbb{R}^+$ and n is defined as in (1). For a wide class of functions, these are equivalent [1]. Unfortunately, the Riemann–Liouville fractional derivative is not suitable to be used in the Laplace transform domain since it requires the knowledge of the non-integer order derivatives of the function at $t=0$. This ambiguity does not exist for the Caputo definition and the Laplace transform of the Caputo fractional derivative is defined by

$$\mathcal{L}\{_C D_t^a f(t)\} = s^a \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{a-1-k} f^k(0), \tag{5}$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace operator. For $a \in \mathbb{N}$ this relation reduces to the Laplace transform of the ordinary integer order derivatives. This property makes the definition more suitable. For instance, some important concepts of the classical control theory can be extended to fractional control theory via using the Caputo fractional derivative. For this reason this definition is used in this paper.

After definition of fractional derivative, one can define fractional order differential equations in the state space similar to integer order equations. Consider the following fractional order system:

$$\frac{d^{a_i} x_i}{dt^{a_i}} = f_i(x_1, \ldots, x_N), \quad i = 1, 2, \ldots, N, \tag{6}$$

where $a_i = n_i/m_i \in \mathbb{Q}^+$ and $f_i(\cdot)$'s are continuous first order differentiable functions (note although we consider rational fractional derivatives, the final results of the paper will be extended for $\mathbb{R}^+$). The equilibrium points of (6), $\mathbf{x}^* \in \mathbb{R}^N$, are roots of the following (nonlinear) equations:

$$f_i(x_1, \ldots, x_N) = 0, \quad i = 1, 2, \ldots, N, \tag{7}$$

To study oscillatory behavior of this fractional order system we need to study the stability of the equilibrium points. In the case of integer order systems this can be verified by forming the system Jacobian at equilibrium points and