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Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Robust fractional order differentiators using generalized modulating functions method

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A R T I C L E I N F O

Article history:
Received 8 January 2014
Received in revised form
4 May 2014
Accepted 12 May 2014

Keywords:
Fractional order differentiator
Riemann–Liouville derivative
Generalized modulating functions method
Noise

A B S T R A C T

This paper aims at designing a fractional order differentiator for a class of signals satisfying a linear differential equation with unknown parameters. A generalized modulating functions method is proposed first to estimate the unknown parameters, then to derive accurate integral formulae for the left-sided Riemann–Liouville fractional derivatives of the studied signal. Unlike the improper integral in the definition of the left-sided Riemann–Liouville fractional derivative, the integrals in the proposed formulae can be proper and be considered as a low-pass filter by choosing appropriate modulating functions. Hence, digital fractional order differentiators applicable for on-line applications are deduced using a numerical integration method in discrete noisy case. Moreover, some error analysis are given for noise error contributions due to a class of stochastic processes. Finally, numerical examples are given to show the accuracy and robustness of the proposed fractional order differentiators.

1. Introduction

Fractional calculus was introduced in many fields of science and engineering long time ago. It was first developed by mathematicians in the middle of the ninetieth century [1]. During the past decades, fractional calculus has gained great interest in several applications [2–4]. For instance, fractional derivatives can improve the performances and robustness properties in control design (see, e.g. [5–8]) and in signal processing applications (see, e.g. [9–12]). The fractional order numerical differentiation is concerned with the estimation of the fractional order derivatives of an unknown signal from its discrete noisy observed data. As in the integer order case, this problem is an ill-posed problem in the sense that a small noise can lead to a large error in approximated derivatives. In order to overcome this problem, various robust fractional order differentiators have been proposed in the frequency domain (see, e.g. [13,14]) and in the time domain, such as digital fractional order Savitzky–Golay differentiator [15], fractional order Jacobi differentiator [16], and B-Spline functions-based fractional order differentiator [17]. The main idea of the latter fractional order differentiators designed in the time domain is to use a polynomial to approximate the unknown signal whose fractional order derivatives are estimated by differentiating the polynomial. If we consider the used polynomial as the truncated Taylor series expansion of the unknown signal, then these fractional order differentiators contain an estimation error due to the truncated term in the Taylor series expansion. When estimating the fractional order derivatives of an unknown signal, even in noise free case, these kinds of truncation errors can produce large errors near the boundaries of the interval where the fractional order derivatives are estimated [16,17].

Existing fractional order differentiators are usually extensions of integer order differentiators [18–20]. When estimating the derivatives of an unknown signal, if the differentiators do not depend on any model which gives the unknown signal, then we call them model-free differentiators. In order to avoid truncation errors, integer order model-based differentiators have been proposed using

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http://dx.doi.org/10.1016/j.sigpro.2014.05.016
0165-1684/

a recent algebraic parametric method to estimate the state variables of an input–output linear system [21–23]. The successive integer order derivatives of the output were accurately estimated from its noisy observation without any truncation error. The idea of this algebraic parametric method is to apply the Laplace transform to the linear differential equation which defines the studied linear system. By applying some algebraic operations (such as differentiations and multiplications) in the Laplace operational domain, undesired terms in the obtained equation are eliminated. When returning into the time domain, the integer order derivatives of the output are exactly given by integral formulae involving the output and a combination of the weight functions of Jacobi orthogonal polynomials. Then, integer order differentiators are deduced by taking the noisy observation of the output in the obtained integral formulae. It has been shown in [24,25] that thanks to the integral formulae these differentiators exhibit good robustness properties with respect to corrupting noises even if the statistical properties of the noises are unknown. Recall that the algebraic parametric method was introduced by Fliess and Sira-Ramírez for linear identification [26], and has been extended to many applications in noisy environment, such as design of integer order model-free differentiators (see, e.g. [27–31]) and parameter estimation (see, e.g. [32–37]). Very recently, it has been applied for fractional order model-free differentiators [38] and for fractional order systems identification [39]. However, the algebraic parametric method has not been applied for fractional order model-based differentiators.

Modulating functions method introduced by Shinbrot [40] is very similar to the algebraic parametric method. This method has been widely used for linear and non-linear identification of continuous-time systems (see, e.g. [41–43]), and parameter estimation of noisy sinusoidal signals (see, e.g. [44,36]). The idea of this method is to multiply a class of modulating functions to a linear differential equation of the analyzed signal. Then, an integration over a finite interval is taken to the obtained equation. Application of integration by parts allows us to remove the derivative operations from the analyzed signal to the multiplied modulating functions, and the undesired boundary values are eliminated thanks to the properties of modulating functions. Finally, estimators are obtained by solving a linear system of algebraic equations and given by integral formulae involving the noisy observation of the analyzed signal. Since the weight functions of Jacobi orthogonal polynomials obtained in integral formulae by the algebraic parametric method are also a class of modulating functions, the modulating functions method can be considered as a generalization of the algebraic parametric method in some cases (see, e.g. [36]). It has similar advantages to the algebraic parametric method, especially the robustness properties with respect to corrupting noises. Moreover, inspired by the algebraic parametric method, the modulating functions method can be extended to many applications. In [45], a fractional integration by parts formula has been obtained by working in the operational domain, then the modulating functions method has been generalized to fractional order systems identification problem. Inspired by [21–23], generalized modulating functions have been given [46], whose existence can be guaranteed by the algebraic parametric method. Then, the modulating functions method has been generalized to design an integer order model-based differentiator [46], where the proposed differentiator does not contain any truncation error. Unlike the integer order model-based differentiators obtained with complex mathematical deduction in [22,23], this differentiator is easy to obtain and to understand. Having these ideas in mind, the aim of this paper is to extend the modulating functions method to design a robust fractional order model-based differentiator without any truncation error. For this purpose, we will focus on a specific class of signals satisfying a linear differential equation with unknown parameters.

This paper is organized as follows. Section 2 begins with some basic definitions of fractional order derivatives. Then, a recent fractional integration by parts formula is recalled. In Section 3, generalized modulating functions are first proposed. Then, the unknown parameters of the considered linear differential equation are estimated using a set of modulating functions. The generalized modulating functions and the fractional integration by parts formula are applied to obtain fractional order differentiators in continuous noise free case, which provide exact integral expressions for fractional order derivatives. Digital fractional order differentiators are deduced using the obtained integral expressions in discrete noisy case. Moreover, some error analysis results for noise error contribution are given. In Section 4, numerical results illustrate the accuracy and robustness of the proposed fractional order differentiators. Finally, some conclusions and perspectives are given in Section 5. Some classes of generalized modulating functions are presented in Appendix.

2. Preliminary

2.1. Riemann–Liouville and Caputo fractional derivatives

Let $l \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus \mathbb{N}$ with $l-1 < \alpha < l$, and $f \in \mathcal{C}^l(\mathbb{R})$ where $\mathcal{C}^l(\mathbb{R})$ refers to the set of functions being l-times continuously differentiable on $\mathbb{R}$. Then, we recall the following definitions.

Definition 1 (*Podlubny* [3, p. 68]). The left-sided Riemann–Liouville fractional derivative of f is defined as follows: $\forall t \in [a, +\infty[$,

$$ {}_R D^{\alpha}_{a,t} f(t) := \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_a^t (t-\tau)^{l-\alpha-1} f(\tau)\, d\tau, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function (see [47, p. 255]).

Definition 2 (*Podlubny* [3, p. 79]). The left-sided Caputo fractional derivative of f is defined as follows: $\forall t \in [a, +\infty[$,

$$ {}_C D^{\alpha}_{a,t} f(t) := \frac{1}{\Gamma(l-\alpha)} \int_a^t (t-\tau)^{l-\alpha-1} f^{(l)}(\tau)\, d\tau. \tag{2}$$

Definition 3 (*Kilbas et al.* [4, p. 92]). The right-sided Caputo fractional derivative of f is defined as follows: $\forall t \in]-\infty, b]$,

$$ {}_C D^{\alpha}_{t,b} f(t) := \frac{(-1)^l}{\Gamma(l-\alpha)} \int_t^b (\tau-t)^{l-\alpha-1} f^{(l)}(\tau)\, d\tau. \tag{3}$$

Please cite this article as: D.-Y. Liu, T.-M. Laleg-Kirati, Robust fractional order differentiators using generalized modulating functions method, Signal Processing (2014), http://dx.doi.org/10.1016/j.sigpro.2014.05.016