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Stable regions in the parameter space of delays for LTI fractional-order systems with two delays

Q1 Afshin Mesbahi, Mohammad Haeri

Q2 *Advanced Control Systems Lab, Electrical Engineering, Sharif University of Technology, Tehran, Iran*

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ABSTRACT

This paper studies fractional-order systems of retarded type with two independent delays, and determines the stability regions in spaces of delays. In this approach, an auxiliary polynomial is employed to calculate all purely imaginary roots of the characteristic equation of the system on the imaginary axis. Since roots of the characteristic equation are continuous with respect to delays, these purely imaginary roots determine the stability regions in delay space. Also, the necessary and sufficient condition for stability independent of delays is developed for the systems. Furthermore, a simple inequality constraint is established to obtain pure imaginary poles of the scalar systems. Finally, the obtained results are illustrated by two examples, and the method is applied to analyze the model of human immunodeficiency virus type 1 infection.

1. Introduction

Time delays occur through almost all types of natural systems, because material and energy cannot be instantaneously transmitted [1,2]. Stability analysis of time delay systems is complex due to the non-deterministic polynomial-time hard (NP-hard) nature of the stability problem [3]. An increasing interest has currently turned towards fractional-order differential equations. Fractional-order differential equations are usually employed for the purpose of accurate representations of the memory and hereditary properties of natural phenomena. So, fractional-order systems describe behavior of real physical systems more truthfully than the differential equations which are involved only integer order derivatives [4,5].

Stability of linear fractional-order systems has been exhaustively studied [6,7]. However, considering time delay in the systems causes some difficulties for stability analysis because their characteristic equation involves exponential type transcendental terms as well as non-integer orders. Their characteristic equation is a multi-valued function that has infinite roots. Fractional delay systems of retarded types are the bounded-input bounded-output (BIBO) stable if and only if all roots of their characteristic equation lie in the left of the imaginary axis in the complex plane [8]. For scalar systems, the location of characteristic roots of the systems can be determined by using the Lambert W function [9]. The delay margin of fractional delay systems of retarded type has been obtained by using the Orlando formula [10]. A numerical algorithm based on Cauchy's integral theorem has investigated the stability of fractional delay systems with a constant delay [11]. By employing a generalized form of Hassard's theorem, an analytical criterion is derived to determine the number of unstable roots of the characteristic equation for each given constant value of the delay [12]. Consequently, by using this method, the stability of the systems is determined as well. Based on numerical algorithm, locations of all unstable poles of the system are computed and then the stability of the system is determined [13]. However, it is assumed that the root paths at the purely imaginary characteristic roots cannot be tangent to the imaginary axis. Recently, the algebraic stability test procedure has been

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motivated to overcome this restriction [14]. This technique divides the space of delay parameters into distinct regions where the system is either stable or unstable for all values of delay in each region. The stability analysis of these systems can be yielded by drawing the value sets [15]. However, the aforementioned method must be separately applied for each point in the delay space to determine stability regions of delay values. So, the boundaries between the stability regions in the space of time delays cannot be accurately calculated by the method in Ref. [15].

Two independent delays should be considered as an intrinsic property of many natural and artificial systems [16–18]. Stability analysis of systems with two independent delays is far more complicated than with one delay. For a scalar system, the boundary region of delays including the origin has been obtained by calculating characteristic roots of systems on the imaginary axis [19]. The root locus crossing directions at these positions depend only on the crossing frequency and are invariant with respect to delay value [20,21]. This argument has been used to check delay-independent stability of systems with multiple delays [22]. The methods mentioned so far are employed to locate the poles of TDS with respect to the imaginary axis [19–22]. However, the stability of fractional-order systems is obtained with respect to two straight lines passing through the origin in a new complex plane [23]. Therefore, a modification of the existing methods is required to analyze the stability of fractional delay systems.

Roots of the characteristic equation on the imaginary axis play an important role in determining the region of other roots. Therefore, all pairs of purely imaginary roots of the characteristic equation are calculated for all admissible values of time delay. This is performed by transforming the domain of the systems from a multi-sheeted Riemann surface into another complex plane and by substituting the exponential terms with a rational function, which have been respectively applied for fractional delay systems in Refs. [23] and [24,25]. The mentioned locations on the imaginary axis are called crossing points. These crossing points divide the space of delays into countable regions. To determine the BIBO-stability of the system for every region of delays, an arbitrary point in the region is selected. Then the origin of the space is connected to the picked point with an arbitrary trajectory. Intersections between the trajectory and the boundaries separating the regions are computed. Then, the number of unstable roots for every point in this trajectory is determinable based on the direction of the root path at every intersection and the root continuity argument. The number of crossing points on the imaginary axis is finite for the systems with one delay; however it can be infinite for systems with two independent delays. For scalar systems, a simple inequality constraint is derived to compute all crossing points. Additionally, the values of delays which cause these crossing points are analytically calculated.

This paper is organized as follows. In Section 2, definitions, preliminary facts, and assumptions are introduced which are used later on in this paper. The stability condition of fractional-order delay systems is described in Section 3. The method of analyzing the BIBO-stability of the fractional-order delay systems is explained in Section 4 and an analytical solution for scalar systems is presented. In Section 5, two examples are provided to explain the obtained results. In addition, the stability of HIV-1 infection systems, described by fractional-order differential equations including two time delay terms, are investigated to illustrate the effective of the developed lemma and theorems. Finally Section 6 concludes the paper.

2. Problem formulation and assumptions

Let ε, j, and $\mathrm{Arg}(z)$ denote an arbitrarily small positive real number, $\sqrt{-1}$, and the principal value of the argument of z assuming $z \neq 0$ (i.e., $\mathrm{Arg}(z) \in (-\pi, \pi]$). Define sets $\mathbb{R}^+ = \{z \in \mathbb{R} : z > 0\}$ and $A \setminus B = \{z \in A : z \notin B\}$. Consider a commensurate fractional-order linear time invariant system (LTI) with two independent delays that are governed by the following state space equation.

$$D^{\alpha}\boldsymbol{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau_1) + \mathbf{C}\mathbf{x}(t-\tau_2), \tag{1}$$

where $\mathbf{A}$, $\boldsymbol{B}$, and $\mathbf{C}$ are given $n \times n$ real constant matrices, $0 < \alpha < 1$, n is an integer number, $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, τ_1 and τ_2 are non-negative real numbers, and D^{α} denotes the Caputo derivative. The characteristic equation of system (1) is

$$\mathrm{CE}(s, \tau_1, \tau_2) = \det(s^{\alpha}\mathbf{I}_n - \mathbf{A} - \mathbf{B}e^{-\tau_1 s} - \mathbf{C}e^{-\tau_2 s}). \tag{2}$$

In a more general form, (2) can be considered as follows:

$$\mathrm{CE}(s, \tau_1, \tau_2) = s^{n\alpha} + \sum_{m=0}^{n-1}\sum_{k=0}^{n}\sum_{l=0}^{n} a_{mkl} e^{-(k\tau_1 + l\tau_2)s} s^{m\alpha}, \tag{3}$$

where $a_{mkl} \in \mathbb{R}$. Notice that the highest order of s in the characteristic equation is not influenced by the delay values (τ_1 or τ_2).

Assumption 1. The real parts of all eigenvalues of $\mathbf{A}+\mathbf{B}+\mathbf{C}$ are nonzero.

This assumption means that the characteristic equation of the delay-free system does not have any root on the imaginary axis. Due to Assumption 1, the origin in the complex plane would be a root of characteristic equation (3) only if either one or two of the delays becomes infinity. The assumption also excludes stationary roots (those that do not depend on delays values) at $s = 0$.

Assumption 2. The characteristic equation does not have high order roots on the imaginary axis. As a result of this assumption, all roots of the characteristic equation on the imaginary axis are simple.

3. Stability analysis of fractional delay system

A worthwhile test for stability analysis of fractional delay systems is achieved by a suitable change of variable as $v = s^{\alpha}$. Since the fractional order is positive and less than one, this mapping transforms the domain of system from a multi-sheeted Riemann surface into the principal sheet of the Riemann surface. Therefore, roots of characteristic equation (3) can be easier to calculate. By applying this substitution to (2), the quasi-characteristic equation of the

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