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Unified approach to extrapolation of bandlimited signals in linear canonical transform domain

Hui Zhao [a,b,*], Ruyan Wang [a], Daiping Song [c], Tianqi Zhang [a,b], Yuanni Liu [a]

[a] School of Communication and Information Engineering, Chongqing University of Posts and Telecommunications, Chongqing 400065, China
[b] Chongqing Key Laboratory of Signal and Information Processing, Chongqing 400065, China
[c] College of Mechanical Engineering, Chongqing University, Chongqing 400044, China

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ABSTRACT

The linear canonical transform (LCT) has been shown to be a powerful tool for signal processing and optics. Several extrapolation strategies for bandlimited signals in LCT domain have been proposed. The purpose of this paper is to present an approach that unifies a number of different algorithms for the extrapolation of bandlimited signals in LCT domain. This unification is achieved through integral equation and Hilbert space theories. First, the following existing techniques are unified: (1) a continuous signal extrapolation algorithm based on series expansion in terms of generalized prolate spheroidal functions; (2) a generalized Papoulis–Gerchberg iterative algorithm; (3) a two-step extrapolation algorithm for continuous signal from finite samples; and (4) an iterative extrapolation algorithm based on error energy reduction procedure for continuous signal from finite samples. Then, two extrapolation algorithms for discrete bandlimited signals in LCT domain are proposed, which also belongs to the unified framework.

1. Introduction

The linear canonical transform (LCT) [1,2], which is also known as the ABCD transform [3], the affine Fourier transform [4] and the generalized Fresnel transform [5], is a four-parameter class of linear integral transforms. It includes the conventional Fourier transform, the fractional Fourier transform, the Fresnel transform, simple scaling and chirp multiplication operations as special cases [6]. Free space propagation in the Fresnel approximation, transmission through thin lenses, propagation through quadratic graded-index media, and their arbitrary combinations fall into the first-order optical systems or quadratic-phase systems, which are mathematically equivalent to the LCTs [6]. Because of the extra degrees of freedom, LCT is more flexible and has been proved to be one of the most powerful tools for signal processing, optics, etc. [7–11].

Bandlimited signal extrapolation is a fundamental problem in signal processing and communication [12–17]. Associated with LCT, many extrapolation strategies for bandlimited signals in LCT domain have been proposed. These works go back to [18], where Sharma and Joshi generalized the Fourier transform based Gerchberg–Papoulis algorithm to the case of fractional bandlimited signals by projecting the iterations alternately in the time and fractional domains. Since then, several algorithms for bandlimited signal extrapolation associated with LCT have appeared, such as extrapolation algorithms for continuous signals from continuous segments [19–21], extrapolation algorithms for continuous signals from finite samples [22],

* Corresponding author at: School of Communication and Information Engineering, Chongqing University of Posts and Telecommunications, Chongqing 400065, China. Tel.: +86 023 62461473; fax: +86 023 62461473.
E-mail address: zhaohui@cqupt.edu.cn (H. Zhao).

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and extrapolation algorithms for discrete bandlimited signals in LCT domain [23].

Motivated by the unification of linear signal restoration [24], the purpose of this paper is to describe a unified framework for the extrapolation algorithms for bandlimited signals in LCT domain. First, the following existing extrapolation algorithms with different underlying models are unified:

(a) An algorithm based on series expansion in terms of generalized prolate spheroidal wave functions (GPSWFs), which is used for continuous signal extrapolation from continuous segment [19].
(b) A generalized Papoulis–Gerchberg iterative algorithm for continuous signal extrapolation from continuous segment [18,21].
(c) A two-step extrapolation algorithm for continuous signal from finite samples [22].
(d) An iterative extrapolation algorithm based on error energy reduction procedure for continuous signal from finite samples [22].

Then, two extrapolation algorithms for discrete bandlimited signals in LCT domain are proposed. It can be shown that the proposed algorithms also belong to the unified framework. Specially, all these algorithms are derived by considering the solution of linear operations in Hilbert spaces.

The outline of this paper is organized as follows. In the next section, some preliminaries that will be used in this paper are presented. In Section 3, the extrapolation problem of continuous bandlimited signal in LCT domain from continuous segment is discussed. The extrapolation problem is posed as a Fredholm integral problem of the first kind and then, the reviewed results given in Section 2 are used to derive the algorithms (a) and (b) as byproducts. In Section 4, the extrapolation problem of continuous bandlimited signal in LCT domain from finite samples is discussed. The extrapolation problem is shown to be equivalent to solving optimization problems in Hilbert spaces and then, the reviewed results given in Section 2 are used to derive the algorithms (c) and (d) as byproducts. In Section 5, the extrapolation problem of discrete bandlimited signal in LCT domain is discussed. And by the reviewed results given in Section 2, two extrapolation algorithms are proposed. Conclusions and discussions appear in Section 6.

2. Preliminaries

2.1. The LCT

2.1.1. The LCT of continuous signal

The LCT with parameters $\{a,b,c,d\}$, or briefly (a, b, c, d)-LCT of a finite energy continuous signal f is defined as [6]

$$F_{(a,b,c,d)}(u)=(\mathcal{L}_{(a,b,c,d)}f)(u) = \begin{cases} \int_{-\infty}^{\infty} f(t)\mathcal{K}_{(a,b,c,d)}(t,u)\,dt, & b\neq 0 \\ d^{1/2}e^{icdu^2/2}f(du), & b=0 \end{cases} \tag{1}$$

with

$$\mathcal{K}_{(a,b,c,d)}(t,u)=\sqrt{\frac{1}{i2\pi b}}e^{((id/2b)u^2)}e^{-((i/b)ut)}e^{((ia/2b)t^2)}. \tag{2}$$

a, b, c, d are real numbers satisfying $ad-bc=1$. The condition $ad-bc=1$ yields that there are only three freedoms a, b and d in the expression (2). The inverse of the LCT with parameters $\{a,b,c,d\}$ is given by a LCT with parameters $\{d,-b,-c,a\}$, that is

$$f(t)=(\mathcal{L}^{-1}_{(a,b,c,d)}F_{(a,b,c,d)})(t) = \begin{cases} \int_{-\infty}^{\infty} F_{(a,b,c,d)}(u)\mathcal{K}_{(d,-b,-c,a)}(u,t)\,du, & b\neq 0 \\ a^{1/2}e^{-icat^2/2}f(at), & b=0, \end{cases} \tag{3}$$

where

$$\mathcal{K}_{(d,-b,-c,a)}(u,t)=\overline{\mathcal{K}}_{(a,b,c,d)}(t,u) \tag{4}$$

with superscript denoting complex conjugation. Note that when $b=0$, the LCT is essentially a chirp multiplication. Therefore, from now on we shall confine our attention to LCT for $b\neq 0$. And without loss of generality, we assume that $b>0$ in the following sections.

It can be easily derived that the LCT of continuous signal preserves the energy of the signal. That is, the following Parseval's formula holds:

$$E=\int_{-\infty}^{\infty}|f(t)|^2\,dt=\int_{-\infty}^{\infty}|F_{(a,b,c,d)}(u)|^2\,du, \tag{5}$$

where E is the energy of the signal f.

Let σ be a positive number, if the (a, b, c, d)-LCT $F_{(a,b,c,d)}(u)$ of a continuous signal f vanishes for $|u|>\sigma$, we say that f is σ bandlimited in (a, b, c, d)-LCT domain; or briefly, we say that f is (a, b, c, d)-bandlimited to σ.

2.1.2. The LCT of discrete signal

For any finite energy discrete signal $\mathbf{f}=\ldots, f[-1], f[0], f[1], \ldots$, its (a, b, c, d)-LCT is defined as [25]

$$\mathcal{F}_{(a,b,c,d)}(u)=\sum_{n=-\infty}^{\infty} f[n]\mathcal{K}_{(a,b,c,d)}(n,u). \tag{6}$$

And $\mathbf{f}$ can be represented by its (a, b, c, d)-LCT as

$$f[n]=\int_{-\pi b}^{\pi b}\mathcal{F}_{(a,b,c,d)}(u)\overline{\mathcal{K}}_{(a,b,c,d)}(n,u)\,du. \tag{7}$$

Associated with the LCT of discrete signal, the following Parseval's formula holds:

$$E=\sum_{n=-\infty}^{\infty}|f[n]|^2=\int_{-\pi b}^{\pi b}|\mathcal{F}_{(a,b,c,d)}(u)|^2\,du, \tag{8}$$

where E is the energy of the signal f.

Let $\sigma\le\pi b$ be a positive number. If the (a, b, c, d)-LCT $\mathcal{F}_{(a,b,c,d)}(u)$ of a discrete signal $\mathbf{f}$ vanishes for $|u|>\sigma$, we say that $\mathbf{f}$ is σ bandlimited in (a, b, c, d)-LCT domain; or briefly, we say that $\mathbf{f}$ is (a, b, c, d)-bandlimited to σ.

2.2. Hilbert spaces and operator theory

Let H_1 and H_2 be two Hilbert spaces and $A: H_1\to H_2$ a linear operator. We say that A is bounded if there exists a real number c, such that $\|Ax\|_2\le c\|x\|_1$ for all $x\in H_1$, where $\|\cdot\|_k, k=1,2$ denotes the norm in H_k. If A is a