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Block-sparse recovery via redundant block OMP

Yuli Fu*, Haifeng Li, Qiheng Zhang, Jian Zou

School of Electronic and Information Engineering, South China University of Technology, Guangzhou 510640, China

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ABSTRACT

Recently, it has been found that the redundant blocks problem existed in many fields, such as face recognition and motion segmentation. In this paper, taking the redundant blocks into account, we propose some greedy type algorithms that exploit the subspace information of the redundant blocks to solve the redundant blocks problem. The exact recovery conditions of these algorithms are presented via block restricted isometry property (RIP). Numerical experiments demonstrate the validity of these algorithms in solving the problems with both non-redundant and redundant blocks.

1. Introduction

The sparse signal representation is an important method in signal processing. Its applications include various fields, such as compressed sensing/sampling, computer vision, image reconstruction, blind source separation and so on [1–6]. Typically, for a given signal $\mathbf{y}$, the representation is modeled as an underdetermined equation

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m \times N}$ denotes a dictionary with $m < N$. The sparse representation model (1) is considered as the one with single measurement vector (SMV). Usually, sparsity in some sense is used as the constraint of the solution to guarantee the uniqueness.

To solve Eq. (1), greedy algorithms are usually used due to their low complexity and simple interpretation in geometry. Theoretically, Tropp [7] presented some conditions to guarantee that orthogonal matching pursuit (OMP) can exactly recover sparse vector $\mathbf{x}$ from (1), using mutual coherence $\mu = \max_{i,j} |\langle \mathbf{A}_i, \mathbf{A}_j \rangle|$, where $\mathbf{A}_i$ denotes the ith column of the dictionary $\mathbf{A}$. Another useful tool for the theoretical analysis is restricted isometry property (RIP). A matrix $\mathbf{A}$ satisfies the RIP of order k if there exists a constant $\delta \in (0, 1)$ such that

$$(1-\delta)\|\mathbf{x}\|_2^2 \le \|\mathbf{A}\mathbf{x}\|_2^2 \le (1+\delta)\|\mathbf{x}\|_2^2 \tag{2}$$

for any k-sparse vector $\mathbf{x}$ that has at most k non-zero elements. In particular, the minimum of all the constants δ satisfying (2) is defined as restricted isometry constant (RIC) δ_k.

Note that there exist matrices satisfying RIP condition but not the mutual coherence condition [8]. It is a motivation to search the exact recovery condition of OMP using RIP. Employing RIP, a bound was given by [8] to guarantee the exact recovery of OMP. And, it was relaxed by [9,10].

Recently, block-sparse representation model have attracted significant attention. In this model, the vector $\mathbf{x}$ in (1) exhibits additional structure in the form of the non-zero coefficient occurring in blocks (or clusters). In practice, the block-sparse structure can be found in many fields, such as reconstruction of multi-band signals [11], and face recognition [12]. Theoretically, the exact recovery of unknown vectors with block structure was considered in [13–16]. Meanwhile, block OMP (BOMP) algorithm was presented and analyzed based on block-coherence in [14].

The works mentioned above discussed the dictionaries with non-redundant blocks. That is, the vectors in any one block are linearly independent. Contrarily, a block is called a redundant block if it consists of linearly dependent

* Corresponding author. Tel.: +86 20 87114702 602;
fax: +86 20 87114702 603.
E-mail address: fuyuli@scut.edu.cn (Y. Fu).

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vectors. The dictionary with redundant blocks may be found in many fields, such as signal/image processing, machine learning, and computer vision. Sometimes, one block in the dictionary is redundant due to the fact that the number of the data in this block exceeds the dimension of the underlying subspace. Elhamifar and Vidal [17] considered the redundant blocks by convex optimization to obtain the uniqueness condition of the block-sparse representation for a given signal.

Besides the applications, the basic theory of the redundant case, such as the exact recovery condition of a certain algorithm, has been gained much attention. It is found that some theory established for the non-redundant case, cannot be used in this redundant case directly. Since the computational complexity is considered, the greedy type algorithms are used in this paper to solve the problem with redundant blocks.

The multiple measurement vectors (MMV) model arises in various applications including estimation of sparse brain regression [18], multivariate regression [19], and direction of arrival (DOA) estimation [20]. Since the correlated information across the measurements is employed, MMV-based methods have high efficiency and good performance [21–23]. For video processing, Majumdar and Ward [24] presented an MMV-based method. However, the block structure of the dictionary was not considered in their work.

The main contributions of this paper are summarized as follows. First, consider the structure of redundant blocks, the BOMP algorithm is extended to an algorithm for redundant blocks. We term this algorithm BOMPR. The exact recovery conditions of BOMP and BOMPR are presented based on block RIP. Second, by MMV, an algorithm is proposed to solve problems in practice with redundant blocks. It is referred to as BMMVR. The exact recovery condition of BMMVR is presented too. The BMMVR algorithm processes multiple samples simultaneously, and takes the redundant blocks into account as well. The experiments of the face recognition are made to show that BMMVR can obtain high classification rate.

The rest of the paper is organized as follows. In Section 2, we introduce BOMPR and provide the exact recovery conditions of BOMP and BOMPR. In Section 3, we present BMMVR and discuss the exact recovery condition of it based on block RIP. In Section 4, several experiments are made to illustrate the validity of BOMPR and BMMVR. Finally, the conclusion is given in Section 5.

The notations used in this paper are listed here. We denote vectors by boldface lowercase letters, e.g., $\mathbf{x}$, and matrices as boldface uppercase letters, e.g., $\mathbf{A}$. By x_i and $\mathbf{x}[i]$, we denote the ith entry and the ith block of $\mathbf{x}$, respectively. Symbol $\mathbf{A}[i]$ denotes the ith block of $\mathbf{A}$. $\mathbf{A}_i$ denotes the ith column of $\mathbf{A}$. $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ denote the transpose of $\mathbf{A}$ and Moose–Penrose pseudo-inverse of $\mathbf{A}$, respectively. $\mathbf{A}^{r\rightarrow}$ means the rth row of $\mathbf{A}$. $\mathbf{A}_{m\times N}$ is equivalent to $\mathbf{A}\in\mathbb{R}^{m\times N}$. $|\Gamma|$ designates the cardinality of a finite set Γ. S_i is the subspace spanned by the columns of the block $\mathbf{A}[i]$. The standard Euclidean norm is $\|\mathbf{x}\|_2^2=\langle\mathbf{x},\mathbf{x}\rangle=\sum_i|x_i|^2$. The Frobenius norm of $\mathbf{A}$ is $\|\mathbf{A}\|_F^2=\sum_i\|\mathbf{A}_i\|_2^2$. The spectral norm of $\mathbf{A}$ is denoted by $\rho(\mathbf{A})=\lambda_{\max}^{1/2}(\mathbf{A}^T\mathbf{A})$, where $\lambda_{\max}(\mathbf{B})$ is the largest eigenvalue of the positive-semidefinite matrix $\mathbf{B}$. $\mathbf{I}_m$ is the $m\times m$ identity matrix. We write $\mathbf{A}_\Gamma$ for the column submatrix of $\mathbf{A}$ whose indices are listed in the set Γ.

2. BOMPR: the extension of BOMP

In this section, BOMP is extended to BOMPR that concerns the dictionaries with redundant blocks. Using block RIP, the exact recovery conditions of BOMP and BOMPR are given theoretically.

2.1. Preliminary of block-sparsity

For a dictionary $\mathbf{A}\in\mathbb{R}^{m\times N}$ with $m<N$, the underdetermined equation (1) is considered here with a block structure of $\mathbf{x}$.

In order to emphasize the block structure, the system (1) is rewritten as

$$\mathbf{y}=\mathbf{A}\mathbf{x}_B, \tag{3}$$

where the subscript B denotes the vector with the block structure. To define block-sparsity, $\mathbf{x}_B$ is viewed as a concatenation of blocks $\mathbf{x}_B[i]\in\mathbb{R}^{d_i}$, $i\in\Omega:=\{1,2,\ldots,M\}$,

$$\mathbf{x}_B=[\mathbf{x}_B^T[1]\ \mathbf{x}_B^T[2]\ldots\mathbf{x}_B^T[M]]^T. \tag{4}$$

We also rewrite $\mathbf{A}$ as a concatenation of column-blocks $\mathbf{A}[i]$ of size $m\times d_i$, $i\in\Omega$,

$$\mathbf{A}=[\mathbf{A}[1]\ \mathbf{A}[2]\ldots\mathbf{A}[M]]. \tag{5}$$

Definition 1 (*Eldar and Mishali [13]*). A vector $\mathbf{x}_B\in\mathbb{R}^N$ is called block K-sparse over $\mathcal{I}=\{d_1,\ldots,d_M\}$ if $\mathbf{x}_B[i]$ is nonzero for at most K indices i where $N=\sum_{i=1}^M d_i$.

The support of $\mathbf{x}_B$ is defined as $\mathrm{supp}(\mathbf{x}_B)=\{i|\mathbf{x}_B[i]\neq\mathbf{0}_{d_i\times 1}\}$. Let $\Gamma=\mathrm{supp}(\mathbf{x}_B)$. Obviously, we have $|\Gamma|=K$ and $\Gamma\subseteq\Omega$.

Next, to discuss the basic theory, block RIP is employed.

Definition 2 (*Eldar and Mishali [13]*). The dictionary $\mathbf{A}$ has block RIP over $\mathcal{I}=\{d_1,\ldots,d_M\}$ with parameter $\delta^{\mathcal{I}}\in(0,1)$ if for every $\mathbf{h}_B\in\mathbb{R}^N$ that is block K-sparse it holds that

$$(1-\delta^{\mathcal{I}})\|\mathbf{h}_B\|_2^2\le\|\mathbf{A}\mathbf{h}_B\|_2^2\le(1+\delta^{\mathcal{I}})\|\mathbf{h}_B\|_2^2. \tag{6}$$

The minimum of all constants $\delta^{\mathcal{I}}$ satisfying (6) is defined as the block-RIP constant $\delta_K^{\mathcal{I}}$.

Remark 1. Note that if matrix $\mathbf{A}$ satisfies the block RIP condition (6) with $\delta_K^{\mathcal{I}}\in(0,1)$, the columns of $\mathbf{A}[i]$ are linearly independent for all i. Otherwise, there exists a block 1-sparse $\mathbf{h}_B\neq\mathbf{0}$ such that $\|\mathbf{A}\mathbf{h}_B\|_2=0$. It is obviously a contradiction.

Analogous to [25], we give a lemma via the block RIP condition.

Lemma 1. *For finite sets Γ' and Γ'', let $\mathrm{supp}(\hat{\mathbf{x}}_B)=\Gamma'$ and $\mathrm{supp}(\overline{\mathbf{x}}_B)=\Gamma''$. Here, $\Gamma'\cap\Gamma''=\varnothing$, $|\Gamma'|\le K_1$, and $|\Gamma''|\le K_2$. If $\mathbf{A}$ satisfies the block RIP condition (6) with $\delta_{(K_1+K_2)}^{\mathcal{I}}\in(0,1)$, then we have*

$$|\langle\mathbf{A}\hat{\mathbf{x}}_B,\mathbf{A}\overline{\mathbf{x}}_B\rangle|\le\delta_{(K_1+K_2)}^{\mathcal{I}}\|\hat{\mathbf{x}}_B\|_2\|\overline{\mathbf{x}}_B\|_2. \tag{7}$$