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Brief paper

Constrained reachability and trajectory generation for flat systems ☆

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ABSTRACT

We consider the problem of trajectory generation for constrained differentially flat systems. Based on the topological properties of the set of admissible steady state values of a flat output we derive conditions which allow for an a priori verification of the feasibility of constrained set-point changes. We propose to utilize this relation to generate feasible trajectories. To this end we suggest to split the trajectory generation problem into two stages: (a) the planning of geometric reference paths in the flat output space combined with (b) an assignment of a dynamic motion to these paths. This assignment is based on a reduced optimal control problem. The unique feature of the approach is that due to the specific construction of the paths the optimal control problem to be solved is guaranteed to be feasible. To illustrate our results we consider a *Van de Vusse* reactor as an example.

1. Introduction

The problem of transition between set-points is an important control task. Transition problems are usually approached either by feedback control or via two-degree-of-freedom control schemes. While the first approach might lead to sophisticated feedback controllers with aggressive behavior, the latter combines feedforward inputs, which transfer the system smoothly from one set-point to another one, with typically rather simple feedback structures; cf. Devasia, Chen, and Paden (1996), Graichen, Hagenmeyer, and Zeitz (2005) and Hagenmeyer and Delaleau (2003). Feedforward inputs which ensure nominal set-point transition can be obtained by solving an optimal control problem (Bryson & Ho, 1969; Lee & Markus, 1967) or by using system inversion techniques and/or flatness properties; cf. Devasia et al. (1996), Fliess, Lévine, Martin, and Rouchon (1995), Graichen et al. (2005), Hagenmeyer (2003), Lévine (2009) and Sira-Ramírez and Agrawal (2004). These approaches, however, share the general limitation that nonlinear dynamics and constraints on states and/or inputs are difficult to handle. These difficulties stem from the fact that for constrained nonlinear systems confirming whether one set-point is reachable from another one is usually achieved by computation of an admissible trajectory.

In order to combine optimal control methods with system inversion techniques we focus on the special case of differentially flat systems. Exploiting flatness in the context of dynamic optimization and trajectory generation has been considered previously; see Milam, Mushambi, and Murray (2000), Oldenburg and Marquardt (2002), Petit, Milam, and Murray (2001), Sira-Ramírez and Agrawal (2004) and Suryawan, De Doná, and Seron (2011, 2012). Generally speaking, these works convert an infinite dimensional optimal control problem into a finite dimensional static optimization problem by describing the system evolution via parametrized functions, e.g. splines, in a flat output space. One common restriction of these works is that for nonlinear flat systems subject to input and state constraints, the existence of admissible solutions is in general not guaranteed. In the present contribution we tackle this limitation.

To this end we investigate a reachability condition for flat systems, which allows to confirm set-point reachability a priori, i.e. without explicit computation of admissible solutions. The condition is based on a relation between the constrained reachability of flat systems and the topology of the set of steady state values of a flat output. Similar observations are made but not further investigated in Martin, Murray, and Rouchon (1997) and Rothfuß, Rudolph, and Zeitz (1996). We propose to utilize this relation

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in terms of a two-stage-approach to trajectory generation: (a) the planning of a geometric reference path in a flat output space which connects the set-points, and (b) assigning an admissible dynamic trajectory to this curve. The first step is subject to specific conditions and precomputed while in the second step a small dimensional optimal control problem with strict feasibility guarantee is formulated.

The remainder of the present contribution is structured as follows: In Section 2 we present the problem setting, briefly recall the property of differential flatness, and present the main reachability result. The proof of this result prepares the ground for a two-stage approach to trajectory generation for flat systems which is presented in Section 3. Section 4 considers a nonlinear *Van de Vusse* reactor subject to state and input constraints as an example.

Notation

The image of a set $\mathcal{A} \subset \mathbb{R}^n$ under a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is denoted as $f(\mathcal{A})$. The interior of a compact set $\mathcal{B}$ is written as $\mathrm{int}(\mathcal{B})$. The kth time derivative of a function $r : [0, \infty) \to \mathbb{R}$ is written as $\frac{d^k r(t)}{dt^k}$ or more conveniently $r^{(k)}$. $\mathcal{C}^k$ denotes the set of k-times continuously differentiable functions. The solution at time t of an ODE $\dot{x} = f(x, u)$ starting at $x(0) = x_0$ and driven by an input $u : [0, t] \to \mathbb{R}^m$ is written as $x(t, x_0 \mid u(\cdot))$.

2. Problem statement and constrained reachability result

We consider nonlinear systems of the form

$$\dot{x} = f(x, u), \qquad x(0) = x_0, \tag{1a}$$

$$y = h\left(x, u, \dot{u}, \ldots, u^{(l)}\right). \tag{1b}$$

The states $x \in \mathbb{R}^n$ and inputs $u \in \mathbb{R}^m$ are constrained by simply connected compact sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$. The state constraint set is described as $\mathcal{X} = \{x \in \mathbb{R}^{n_x} \mid c_i^x(x) \le 0, c_i^x \in \mathcal{C}^0, i = 1, \ldots, n_{cx}\}$ and the input constraint is $\mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid c_i^u(u) \le 0, c_i^u \in \mathcal{C}^0, i = 1, \ldots, n_{cu}\}$. The control objective is to generate constraint consistent input and state trajectories as well as a finite time T, such that the system is driven from one set-point $(x_0, u_0) \in \mathcal{X} \times \mathcal{U}$ to another set-point $(x_T, u_T) \in \mathcal{X} \times \mathcal{U}$, whereby

$$0 = f(x_i, u_i), \quad i \in \{0, T\}$$

holds. Formally this can be stated as follows.

Problem 1 (*Constrained Set-Point Transition*). Given system (1), an initial set-point $(x_0, u_0) \in \mathcal{X} \times \mathcal{U}$, and a target set-point $(x_T, u_T) \in \mathcal{X} \times \mathcal{U}$. Compute

(i) a finite transition time $T \in [0, \infty)$;
(ii) and an admissible input signal $u : [0, T] \to \mathcal{U} \subset \mathbb{R}^m$ such that the system trajectory satisfies

$$\forall t \in [0, T] : \quad x(t, x_0 \mid u(\cdot)) \in \mathcal{X}, \tag{2a}$$

$$i \in \{0, T\} : \quad (x(i, x_0 \mid u(\cdot)),\ u(i)) = (x_i, u_i). \tag{2b}$$

Note that the solution to this problem is usually not unique. Part (ii) refers to the general problem of reachability in the presence of constraints on states and inputs. Part (i) requires to transfer the system between the set-points in finite time. Often this problem is tackled as an optimal control problem: either with an a priori choice of the transition time T or formulated as a free-end-time optimal control problem; cf. Bryson and Ho (1969) and Lee and Markus (1967). However, it is in general difficult to verify a priori whether – given system (1), the constraint sets $\mathcal{X}, \mathcal{U}$ and set-points (x_i, u_i), $i \in \{0, T\}$ – Problem 1 is feasible.

To provide sufficient conditions on finite-time set-point reachability we restrict the further considerations to the class of *differentially flat* systems (Fliess et al., 1995).

Definition 1 (*Differentially Flat System*). Consider the system (1a). If there exists a variable $\xi = (\xi_1, \ldots, \xi_{n_\xi})^T$ with $\dim \xi = n_\xi = \dim u = m$, such that the following statements hold at least locally:

(i) The variable ξ can be written as a function of the state variables $x = (x_1, \ldots, x_n)^T$, the input variables $u = (u_1, \ldots, u_m)^T$ and a finite number of time derivatives of the input variables

$$\xi = g\left(x, u_1, \ldots, u_1^{(l_1)}, \ldots, u_m, \ldots, u_m^{(l_m)}\right). \tag{3}$$

(ii) The system variables x and u can be expressed as functions of the variable $\xi = (\xi_1, \ldots, \xi_m)^T$ and a finite number of time derivatives of ξ. Hence there exist maps $\Phi_1 : \mathbb{R}^\kappa \to \mathbb{R}^n$, $\kappa = \sum_{i=1}^m k_i$ and $\Phi_2 : \mathbb{R}^{\kappa+m} \to \mathbb{R}^m$ such that

$$x = \Phi_1\left(\xi_1, \ldots, \xi_1^{(k_1-1)}, \ldots, \xi_m, \ldots, \xi^{(k_m-1)}\right) \tag{4a}$$

$$u = \Phi_2\left(\xi_1, \ldots, \xi_1^{(k_1)}, \ldots, \xi_m, \ldots, \xi_m^{(k_m)}\right). \tag{4b}$$

(iii) The components of ξ are differentially independent; they do not fulfill any differential equation.

Then ξ is called a *flat output* of (1a). Furthermore, (1a) is called a (differentially) flat system.

As is well known the flatness property can be exploited in control tasks such as trajectory generation and set-point changes; see e.g. Fliess et al. (1995), Lévine (2009), Sira-Ramírez and Agrawal (2004). We will utilize flatness to tackle Problem 1. Thus we assume the following.

Assumption 1 (*Flat System*). System (1a) is a differentially flat system and (1b) is one of its flat outputs. Furthermore, the flat parametrizations Φ_1, Φ_2 from (4) are continuous on sufficiently large subsets $\mathcal{I} \subseteq \mathbb{R}^\kappa$, $\mathcal{J} = \mathcal{I} \times \hat{\mathcal{J}} \subseteq \mathbb{R}^{\kappa+m}$ of their domains such that

$$\mathcal{X} \times \mathcal{U} \subseteq \Phi_1(\mathcal{I}) \times \Phi_2(\mathcal{J}) \tag{5}$$

holds.

In essence, this assumption states that flatness and continuity of the parametrizations (4) hold for all $(x, u) \in \mathcal{X} \times \mathcal{U}$. Relaxing this will be discussed in Remark 1.

For brevity of presentation the set

$$\mathcal{S} := \left\{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid f(x, u) = 0\right\} \tag{6}$$

denotes the set of steady states of (1) and the map $\tilde{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$

$$\tilde{h}(x, u) := h(x, u, 0, \ldots, 0) \tag{7}$$

is the output corresponding to stationary inputs of (1). The map $\tilde{\Phi} : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$

$$\tilde{\Phi}(y) = \begin{pmatrix} \Phi_1(y_1, 0, \ldots, 0, y_2, 0, \ldots, y_m, 0 \ldots, 0) \\ \Phi_2(y_1, 0, \ldots, 0, y_2, 0, \ldots, y_m, 0 \ldots, 0) \end{pmatrix} \tag{8}$$

is the stationary version of (4). These notions allow us to define the set of stationary outputs which are consistent with the constraints.

Definition 2 (*Consistent Stationary Outputs*). The sets $\mathcal{Y} \subseteq \hat{\mathcal{Y}} \subset \mathbb{R}^m$

$$\hat{\mathcal{Y}} = \left\{y = \tilde{h}(x, u) \mid (x, u) \in \mathcal{S} \cap (\mathcal{X} \times \mathcal{U})\right\}, \tag{9a}$$

$$\mathcal{Y} = \left\{y = \tilde{h}(x, u) \mid (x, u) \in \mathcal{S} \cap \mathrm{int}(\mathcal{X} \times \mathcal{U})\right\} \tag{9b}$$

are called set of constraint consistent steady state outputs and set of strongly constraint consistent steady state outputs, respectively.