Automatica 50 (2014) 1181–1187

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Stochastic stability of Positive Markov Jump Linear Systems[☆]

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ARTICLE INFO

Article history:
Received 24 April 2013
Received in revised form
5 September 2013
Accepted 10 January 2014
Available online 28 February 2014

Keywords:
Markov Jump Linear Systems
Positive systems
Hybrid systems
Stability

ABSTRACT

This paper investigates on the stability properties of Positive Markov Jump Linear Systems (PMJLS's), i.e. Markov Jump Linear Systems with nonnegative state variables. Specific features of these systems are highlighted. In particular, a new notion of stability (Exponential Mean stability) is introduced and is shown to be equivalent to the standard notion of 1-moment stability. Moreover, various sufficient conditions for Exponential Almost-Sure stability are worked out, with different levels of conservatism. The implications among the different stability notions are discussed. It is remarkable that, thanks to the positivity assumption, some conditions can be checked by solving Linear Programming feasibility problems.

1. Introduction

Markov Jump Linear Systems (MJLS's) are a popular class of stochastic systems that are well suited to describe dynamics characterized by random jumps between subsystems induced by external causes, such as random faults, unexpected events, and uncontrolled configuration changes (Boukas, 2005; Costa, Fragoso, & Marquez, 2005; Costa, Fragoso, & Todorov, 2013). Possible applications of jump systems include fault-tolerant systems (Aberkane, Ponsart, Rodrigues, & Sauter, 2008), networked control (Xiao, Xie, & Fu, 2010), communication networks for multi-agent systems (Meskin & Khorasani, 2009), macroeconomic models (do Val & Basar, 1999), pulp and paper industry (Khanbaghi, Malhamé, & Perrier, 2002), energy systems (Angeli & Kountouriotis, 2012) and epidemiology (Otero, Barmak, Dorso, Solari, & Natiello, 2011). The assumptions that the jumps are governed by an underlying Markov chain and the subsystems are linear make these systems amenable to a thorough theoretical analysis while preserving great flexibility.

It is remarkable that, even in the case of jumps between linear time-invariant subsystems, the fundamental property of state stability is much more involved than in the deterministic case and presents several interesting, sometimes intriguing, facets. Indeed, various notions of stochastic stability can be studied which are not equivalent. *Mean-square stability*, implying asymptotic convergence to zero of the expected squared norm of the state is a classical and widely investigated notion. Necessary and sufficient conditions for mean-square stability of MJLS's are available both in continuous and discrete-time (Boukas, 2005; Costa et al., 2005; Feng, Loparo, Ji, & Chizeck, 1992). Mean-square stability is a particular instance of *δ-moment stability*, where the squared norm is replaced by a generic positive power δ of the norm, see Fang, Loparo, and Feng (1994). However, as pointed out in Kozin (1969), these stability properties, dealing with the convergence of the moments of the state norm, would be better replaced by the direct study of the convergence to zero of almost all the sample paths of the state. As a matter of fact, this last notion, that goes under the name of *almost-sure stability*, is much closer to the practical concerns of the user. The connections between the different stability notions are now well understood, see e.g. Fang et al. (1994), Fang and Loparo (2002). It is known that δ-moment stability with a certain δ implies δ-moment stability with smaller values of δ. Furthermore, almost-sure stability is implied by δ-moment stability for any value of δ. Unfortunately, there do not exist direct and easy-to-check necessary and sufficient conditions for verifying almost-sure stability. The alternative is between relatively simple sufficient conditions, that may be however conservative (Fang, 1997; Fang et al., 1994; Tanelli, Picasso, Bolzern, & Colaneri, 2010), and a randomized criterion related to the average norm contractivity over an interval, that can be made arbitrarily close to necessity at the cost of increased computational burden (Bolzern, Colaneri, & De Nicolao, 2006). For

☆ This paper has been partially supported by the European Commission under the MoVeS project, FP7-ICT-2009-257005. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Mario Sznaier under the direction of Editor Roberto Tempo.

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http://dx.doi.org/10.1016/j.automatica.2014.02.016

this reason, mean-square stability, though more restrictive, may appear more convenient, especially for design purposes, as it can benefit from the availability of necessary and sufficient conditions checkable by standard tools.

In this paper we consider the case in which all the subsystems belong to the class of *linear positive systems* (Farina & Rinaldi, 2000), whose state variables remain nonnegative whenever initialized in the positive orthant. Positive systems are frequently used to describe biological systems (e.g. compartmental models) or population dynamics. Other applications are found in chemical reactions, queue processes, traffic modeling, to mention but a few. The stability properties of deterministic positive systems present peculiar features which simplify the analysis. In particular, asymptotic stability can be assessed using linear Lyapunov functions, see e.g. Farina and Rinaldi (2000). It is therefore natural to investigate whether similar simplifications carry over to the stochastic case of *Positive Markov Jump Linear Systems* (PMJLS's), a study that, to the authors' knowledge, is not available in the literature.

With reference to the notions of mean-square, δ-moment and almost-sure stability, we provide a comprehensive picture of sufficient and/or necessary conditions highlighting the role of positivity both from the theoretical and the computational viewpoint. The key questions addressed in the paper regard the possibility of establishing equivalences between stability notions as well as the possible existence of a stability notion equipped with easy to handle necessary and sufficient conditions but less conservative than mean-square stability. As a further contribution, we work out a number of sufficient conditions for almost-sure stability, discussing their degree of conservatism versus usability.

The paper is organized as follows. After introducing in Section 2 the adopted notation and some useful properties, the class of PMJLS's is presented in Section 3 and the various stability notions are reported in Section 4. The main results of the paper are discussed in Section 5, followed by some numerical examples in Section 6. The paper ends with some concluding remarks.

2. Notation and basic properties

In this paper we will conform with the standard convention of denoting scalar and vectors with small letters and matrices with capital letters. The ith entry of vector x will be indicated as x_i and the (i, j)th entry of matrix A as a_{ij}. Moreover, vectors are usually assumed as column vectors and the suffix $'$ corresponds to vector or matrix transposition. The symbol $\mathbf{1}_n$ denotes the n-dimensional vector with all entries equal to 1. The symbol I_n denotes the identity matrix of order n. In both cases, the suffix n will be omitted when the vector (or matrix) size is clear from the context. The symbol e_i stands for the ith column of the identity matrix (again the dimension will be clear from the context).

A (column or row) vector $x = [x_i] \in \Re^n$ is said to be *positive* if all its entries are strictly greater than 0. In that case, we will say that $x \gg 0$. A vector $x = [x_i] \in \Re^n$ is said to be *nonnegative* if all its entries are greater than or equal to 0. In that case, we will say that $x > 0$. Similar definitions and notation apply when x is either *negative* ($x \ll 0$) or *nonpositive* ($x < 0$). The expressions $x \gg y$, $x > y$, $x \ll y$, $x < y$ indicate that the difference $x - y$ is positive, nonnegative, negative, nonpositive, respectively. To indicate that a square matrix $P \in \Re^{n\times n}$ is *positive definite* (*positive semi-definite*), we will use the symbol $P \succ 0$ ($P \succeq 0$). The notation $P \prec 0$ ($P \preceq 0$) means that P is *negative definite* (*negative semi-definite*).

A square matrix $A = [a_{ij}] \in \Re^{n\times n}$ is said to be *Metzler* if its off-diagonal entries are nonnegative, namely $a_{ij} \geq 0$ for $i \neq j$. For a Metzler matrix A, it is known that its eigenvalue λ with maximum real part is always real and is called the *Frobenius eigenvalue*. The corresponding eigenspace is generated by a positive eigenvector, called the *Frobenius eigenvector*, see e.g. Berman and Plemmons (1994), Farina and Rinaldi (2000). A dynamical linear system described by the differential equation $\dot{x}(t) = Ax(t)$, where A is a Metzler matrix, is called a *positive system* because it enjoys the property that any trajectory starting in the positive orthant remains indefinitely confined in it.

A square matrix is *Hurwitz* if all its eigenvalues lie in the open left half plane. A Metzler matrix is Hurwitz if and only if there exist a vector $c \gg 0$ such that $c'A \ll 0$, see e.g. (Farina & Rinaldi, 2000).

The symbols $\|x\|$ and $\|A\|$ will be used to denote a generic *norm* of vector $x \in \Re^n$ and the corresponding *induced norm* of matrix $A \in \Re^{n\times n}$. In particular $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$.

The *measure* of matrix $A \in \Re^{n\times n}$ is defined as $\mu(A) = \lim_{h\to 0} \frac{\|I+Ah\|-1}{h}$ and it depends on the adopted matrix norm. It is well known that $\|\exp(At)\| \leq \exp(\mu(A)t)$, $t \geq 0$. It can be shown that $\mu_1(A) = \max_j \left(a_{jj} + \sum_{i=1,\, i\neq j}^n |a_{ij}|\right)$. These results can be found in Desoer and Vidyasagar (1975).

Given a set of vectors $z_i \in \Re^n$, $i = 1, 2, \ldots, N$, the symbol $v_z = \mathrm{vec}[z_i]$ represents the vector obtained by stacking vectors $z_1, z_2, \ldots, z_N$, into a single nN-dimensional vector. For two matrices $A \in \Re^{n\times m}$, $B \in \Re^{p\times q}$, the expression $C = A \otimes B$ stands for the usual *Kronecker product*, obtained by orderly collecting the blocks $a_{ij}B$ into the matrix $C \in \Re^{np\times mq}$. For two square matrices $A \in \Re^{n\times n}$, $B \in \Re^{p\times p}$, the *Kronecker sum* is defined as $D = A \oplus B = A \otimes I_p + I_n \otimes B \in \Re^{np\times np}$. Properties of Kronecker operators can be found in Graham (1981).

The expectation of a stochastic variable v will be denoted as $E[v]$. The symbol $\Pr\{\cdot\}$ will be used for the probability of an event.

3. Positive Markov Jump Linear Systems

In this paper the attention will be focused on the class of continuous-time Markov jump linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad t \geq 0 \tag{1}$$

where $x(t) \in \Re^n$, $\sigma(t) \in \mathcal{S} = \{1, 2, \ldots, N\}$, and the matrices $A_i \in \Re^{n\times n}$, $i \in \mathcal{S}$ are Metzler matrices, i.e. real square matrices whose off-diagonal entries are nonnegative. The process $\sigma(t)$ is a time-homogeneous Markov stochastic process with infinitesimal generator $\Lambda \in \Re^{N\times N}$. Precisely, let

$$\Pr\{\sigma(t+h) = j|\sigma(t) = i\} = \lambda_{ij}h + o(h), \quad i \neq j \tag{2}$$

where $h > 0$, and $\lambda_{ij} \geq 0$ is the transition rate from mode i at time t to mode j at time $t + h$. The diagonal entries of Λ are defined as

$$\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$$

so that Λ is a Metzler matrix satisfying $\Lambda\mathbf{1} = 0$. Let τ_k, $k = 0, 1, \ldots, \tau_0 = 0$, be the successive sojourn times between jumps. Then, assuming that after the kth jump the system stays in mode i, from (2) it follows that τ_k is exponentially distributed with parameter $-\lambda_{ii}$. Let $\pi_i(t) = \Pr\{\sigma(t) = i\}$ and $\pi(t) = [\pi_1(t) \ \ldots \ \pi_N(t)]'$. Given an initial probability distribution $\pi(0) = [\pi_{01} \ \ldots \ \pi_{0N}]'$, where $\pi_{0i} := \Pr\{\sigma(0) = i\}$, the time evolution of the probability distribution $\pi(t)$ obeys the differential equation

$$\dot{\pi}(t)' = \pi(t)'\Lambda.$$

Note that $\mathbf{1}'\pi(t) = 1$, $\forall t \geq 0$, i.e. $\pi(t)$ is a unit-sum vector. Moreover, if the Markov process is irreducible (see, e.g. Bremaud, 1998), then, for any $\pi(0)$, $\pi(t)$ converges as $t \to \infty$ to a stationary probability vector $\bar{\pi}$ which is the unique unit-sum Frobenius left eigenvector of Λ associated with the Frobenius–Perron null eigenvalue, Berman and Plemmons (1994). In the sequel, it is assumed that the Markov process is irreducible. Moreover, the symbol $E_{\bar{\pi}}[\cdot]$