Automatica 48 (2012) 3135–3141

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

On the stability and stabilization of quaternion equilibria of rigid bodies ☆

Rune Schlanbusch [a], Antonio Loria [b,1], Per Johan Nicklasson [a]

[a] Department of Scientific Computing, Electrical Engineering and Space Technology, Narvik University College, PB 385, AN-8505 Narvik, Norway

[b] CNRS, LSS-Supelec, 3 Rue Joliot Curie, Gif sur Yvette, France

ARTICLE INFO

Article history:
Received 16 June 2010
Received in revised form
3 April 2012
Accepted 5 July 2012
Available online 12 September 2012

Keywords:
Quaternions
Output feedback
Density functions
Rigid body

ABSTRACT

We study attitude control of rigid bodies on quaternion coordinates under three mathematically different perspectives, depending on how the system dynamics are assumed to evolve. In the first case, we suppose that one equilibrium point is chosen *a priori* and a continuous controller is used under the assumption that the rigid body always spins in the same direction. In the second case, we relax the assumption that the sense of rotation is constant. Finally, a third scenario is considered in which hybrid (switching) control is used to *choose* the direction in which to spin, that is, both equilibria are continuously considered with regard to less energy consumption. It is showed that each of three scenarios must be treated in a different theoretical setting. A comparative study in simulations is also provided.

1. Introduction

The attitude of a rigid body may be described by so-called quaternions, *redundant* coordinates on the space $SO(3)$. Quaternions determine any point on the sphere and include one "extra" coordinate which indicates the sense of rigid-body rotations. They are redundant as the two poles of the sphere correspond to the same physical posture of the body yet, mathematically they account for two equilibria. This brings especial difficulties to the stability analysis of attitude-controlled rigid bodies.

From a practical viewpoint, certain control actions may cause the body to rotate almost a full revolution to achieve a posture which is close to the initial one, *i.e.*, to take a longer path. From an analytical view-point the two equilibria must be considered as different hence, one may not expect to achieve "global" stability properties in closed-loop. Besides, the adjective "global" or "in the whole" pertains to the case when the states are elements of $\mathbb{R}^n$—see Hahn (1967). See also Loría and Panteley (2006) for precise definitions of stability and discussions.

To deal with multiple equilibria in control design there are two evident alternatives: to choose a target equilibrium before starting a maneuver, or not. If a target equilibrium is fixed before the maneuver the control design relies on the hypothesis that the sense of rotation does not change. Mathematically, this is tantamount to assuming that one of the quaternion states does not change sign. In Kristiansen, Nicklasson, and Gravdahl (2008) the authors proposes a controller which steers a spacecraft to the equilibrium point closest to the initial posture. However, the shortest-path rotation is not necessarily optimal in terms of use of input "energy"—for instance, fuel consumption in the context of spacecraft control, if initial velocities are relatively high and in direction opposite to the desired rotation. See Schlanbusch, Kristiansen, and Nicklasson (2010a) for a study of this aspect.

The freedom of not fixing the reference equilibrium *a priori* comes at the price of the increased complexity. See for instance Casagrande, Astolfi, and Parisini (2008) on control of an under-actuated non-symmetric rigid body and Mayhew, Sanfelice, and Teel (2009) where the authors present two quaternion-based *hybrid* controllers: one is derived from an energy-based Lyapunov function which entails a switch in the rotational direction only when the rotational error is above π rad and one based on backstepping design which also considers the angular velocity errors to determine whether switching is needed.

In this paper we analyze the three following scenarios

Scenario 1.— One equilibrium is considered and is chosen a priori;
Scenario 2.— Two equilibria are considered, one of which is chosen a priori;
Scenario 3.— Two equilibria are considered, none of which is chosen a priori.

For comparison, we use in the three cases a controller that is inspired from Slotine and Li (1988). However, the controller that

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Alessandro Astolfi under the direction of Editor Andrew R. Teel.

E-mail addresses: runsch@hin.no (R. Schlanbusch), loria@lss.supelec.fr (A. Loria), pjn@hin.no (P.J. Nicklasson).

[1] Tel.: +33 1 69 85 1724; fax: +33 1 69 85 1765.

0005-1098/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2012.08.012

we propose is adapted to the rigid body in quaternion space hence, it is different from that in the latter reference, which applies to robot manipulators in joint space ($\mathbb{R}^{2n}$).[2] In the first case, the controller is showed to guarantee asymptotic stability *in the large* provided that the sense of rotation is constant. In the second case, we add to the previous controller a switching rule which yields only one potential *initial* switch and we provide a proof of asymptotic stability using density functions—see Rantzer (2001). We show asymptotic stability for all initial conditions on the sphere except for a zero-measure set. In the third case we incorporate a switching law, the closed-loop system is *hybrid* and we use the framework of Sanfelice, Goebel, and Teel (2007) to analyze the closed-loop system.

Simulation results are presented with the aim at showing that a proper design of the switching law may lead to performance improvement.

The rest of the paper is organized as follows. In Section 2 we describe the quaternion-based model of a rigid-body; in Section 3 we present our main results; in Section 4 we present a comparative simulations study and we conclude with some remarks in Section 5.

2. Rigid-body model

Attitude control consists in achieving any rigid-body orientation relative to a fixed frame, independent of that attached to the body itself. Perhaps the best manner to explain the kinematics and dynamics is to consider the attitude of a satellite relative to the Earth. We denoted the body frame as $\mathcal{F}^b$, and is located at the center of mass of the rigid body, and its basis vectors are aligned with the main axis of inertia and the inertia frame as $\mathcal{F}^i$.

2.1. Quaternions

We recall that the special orthogonal group of order three corresponds to the set $SO(3)$ of orthonormal rotation matrices $\mathbf{R}$,

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} : \mathbf{R}^\top \mathbf{R} = \mathbf{I}, \det(\mathbf{R}) = 1\},$$

where $\mathbf{I}$ denotes the identity matrix. A rotation matrix for a rotation θ about an arbitrary unit vector $\mathbf{k}$ may be angle-axis parameterized as in Egeland and Gravdahl (2002), *i.e.*,

$$\mathbf{R}_{k,\theta} = \mathbf{I} + \mathbf{S}(\mathbf{k})\sin(\theta) + \mathbf{S}^2(\mathbf{k})(1 - \cos(\theta)). \tag{1}$$

Then, the coordinate transformation of a vector $\mathbf{r}$ from frame a to frame b is written as $\mathbf{r}^b = \mathbf{R}_a^b \mathbf{r}^a$. The rotation matrix in (1) can be expressed by an Euler parameter representation as

$$\mathbf{R} = \mathbf{I} + 2\eta\mathbf{S}(\boldsymbol{\epsilon}) + 2\mathbf{S}^2(\boldsymbol{\epsilon}),$$

where the matrix $\mathbf{S}(\cdot)$ is the cross product operator, *i.e.*,

$$\mathbf{S}(\boldsymbol{\epsilon}) = \begin{bmatrix} 0 & -\epsilon_z & \epsilon_y \\ \epsilon_z & 0 & -\epsilon_x \\ -\epsilon_y & \epsilon_x & 0 \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{bmatrix}.$$

Quaternions are used to parameterize elements of $SO(3)$. The unit quaternion is defined as $\mathbf{q} = [\eta, \boldsymbol{\epsilon}^\top]^\top \in S^3 = \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{x}^\top\mathbf{x} = 1\}$, where $\eta = \cos(\theta/2) \in \mathbb{R}$ is 'the scalar part' and $\boldsymbol{\epsilon} = \mathbf{k}\sin(\theta/2) \in \mathbb{R}^3$ is 'the vector part'. The set S^3 forms a group with quaternion multiplication, which is distributive and associative, but not commutative. The inverse rotation of a given attitude is performed via the inverse conjugated $\bar{\mathbf{q}} = [\eta, -\boldsymbol{\epsilon}^\top]^\top$.

The difference between two postures is given by the quaternion product,

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = \begin{bmatrix} \eta_1\eta_2 - \boldsymbol{\epsilon}_1^\top \boldsymbol{\epsilon}_2 \\ \eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \mathbf{S}(\boldsymbol{\epsilon}_1)\boldsymbol{\epsilon}_2 \end{bmatrix}.$$

We stress that the quaternion representation is redundant. Notice that $\mathbf{q}$ and $-\mathbf{q}$ represent the *same* physical attitude however, the two postures differ mathematically by a 2π rotation about an arbitrary axis. As a consequence, the mathematical model has two equilibria and this must be considered when studying stability.

2.2. Kinematics and dynamics

The time derivative of the rotation matrix is

$$\dot{\mathbf{R}}_b^a = \mathbf{S}\left(\boldsymbol{\omega}_{a,b}^a\right)\mathbf{R}_b^a = \mathbf{R}_b^a\mathbf{S}\left(\boldsymbol{\omega}_{a,b}^b\right),$$

where $\boldsymbol{\omega}_{a,b}^a \in \mathbb{R}^3$ is the angular velocity of a frame $\mathcal{F}^b$ relative to a frame $\mathcal{F}^a$, expressed in frame $\mathcal{F}^a$. Correspondingly, the kinematic equation is

$$\dot{\mathbf{q}} = \mathbf{T}(\mathbf{q})\boldsymbol{\omega}, \quad \mathbf{T}(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} -\boldsymbol{\epsilon}^T \\ \eta\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}) \end{bmatrix} \in \mathbb{R}^{4\times 3}.$$

The rigid body dynamics is derived from Euler's moment equation, which describes the relation between applied torque and angular momentum on the rigid body, *i.e.*,

$$\mathbf{J}\dot{\boldsymbol{\omega}} = -\mathbf{S}(\boldsymbol{\omega})\mathbf{J}\boldsymbol{\omega} + \boldsymbol{\tau}, \tag{2}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}_{i,b}^b$ is the angular velocity of the body frame $\mathcal{F}^b$ relative to an inertia frame $\mathcal{F}^i$, expressed in the body frame; $\boldsymbol{\tau} \in \mathbb{R}^3$ is the total torque working on the body frame,[3] and $\mathbf{J} \in \mathbb{R}^{3\times 3} = \mathrm{diag}\{J_x, J_y, J_z\}$ is the inertia matrix.

3. Attitude control on quaternion coordinates

The attitude control problem consists in making the actual attitude converge towards a given reference attitude $\mathbf{q}_d$ satisfying the kinematic equation

$$\dot{\mathbf{q}}_d = \mathbf{T}(\mathbf{q}_d)\boldsymbol{\omega}_d.$$

Assumption 1. (a) The desired attitude $\mathbf{q}_d$, the desired angular velocity $\boldsymbol{\omega}_d$ and the desired angular acceleration $\dot{\boldsymbol{\omega}}_d$ are all bounded functions; (b) the desired reference is such that the quaternion errors satisfy the quaternion constraint $\tilde{\boldsymbol{\epsilon}}^\top\tilde{\boldsymbol{\epsilon}} = 1 - \tilde{\eta}^2$.

The quaternion error is given by $\tilde{\mathbf{q}} = \bar{\mathbf{q}}_d \otimes \mathbf{q}$ and yields $\tilde{\mathbf{q}} = [\tilde{\eta}, \tilde{\boldsymbol{\epsilon}}^\top]^\top$ with $\tilde{\eta} \in [-1, 1]$, by definition. The control goal is to steer $\tilde{\boldsymbol{\epsilon}}(t)$ to zero under Assumption 1. Correspondingly, in view of the quaternion constraint, $\tilde{\eta}$ must converge either to $+1$ or to -1.

Now, the error kinematic equation is given by

$$\dot{\tilde{\mathbf{q}}} = \mathbf{T}(\tilde{\mathbf{q}})\mathbf{e}_\omega, \tag{3}$$

where $\mathbf{e}_\omega = \boldsymbol{\omega} - \boldsymbol{\omega}_d$. We remark that due to the redundancy of the quaternion coordinates (3) has two equilibria, which we represent by $(\tilde{\mathbf{q}}_+, \mathbf{e}_\omega) = ([1\,\mathbf{0}], \mathbf{0})$ and $(\tilde{\mathbf{q}}_-, \mathbf{e}_\omega) = ([-1\,\mathbf{0}], \mathbf{0})$ where $\mathbf{0} = [0 \cdots 0]^\top$ is of appropriate dimensions. For the purpose of analysis we translate the problem of stabilizing an equilibrium to that of stabilizing the *origin*. For this, we define the attitude error vector $\mathbf{e}_{q+} = [1 - \tilde{\eta}, \tilde{\boldsymbol{\epsilon}}^\top]^\top$ for the "positive" equilibrium and use $\mathbf{e}_{q-} = [1 + \tilde{\eta}, \tilde{\boldsymbol{\epsilon}}^\top]^\top$ for the "negative" equilibrium. The kinematic relation can then be expressed as

$$\dot{\mathbf{e}}_{q\pm} = \mathbf{T}_e(\mathbf{e}_{q\pm})\mathbf{e}_\omega, \quad \text{where } \mathbf{T}_e(\mathbf{e}_{q\pm}) = \frac{1}{2}\begin{bmatrix} \pm\tilde{\boldsymbol{\epsilon}}^\top \\ \tilde{\eta}\mathbf{I} + S(\tilde{\boldsymbol{\epsilon}}) \end{bmatrix}. \tag{4}$$

In the first two scenarios described below, the control strategy relies on choosing a target equilibrium before the manoeuvre. In the first case, we assume that the body's sense of rotation is

[2] We remark that the choice of the controller is unimportant, *i.e.*, the same results may be obtained for many other controllers inspired from robot control literature.

[3] Typically, in the context of attitude control of spacecraft, $\boldsymbol{\tau}$ contains the control inputs and external disturbances. The latter are not considered in this paper.