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Brief paper

Computable convergence bounds of series expansions for infinite dimensional linear-analytic systems and application☆

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ABSTRACT

This paper deals with the convergence of series expansions of trajectories for semi-linear infinite dimensional systems, which are analytic in state and affine in input. A special case of such expansions corresponds to Volterra series which are extensively used for the analysis, the simulation and the control of weakly nonlinear finite dimensional systems. The main results of this paper give computable bounds for both the convergence radius and the truncation error of the series. These results can be used for model simplification and analytic approximation of trajectories with a guaranteed quality. They are available for distributed and boundary control systems. As an illustration, these results are applied to an epidemic population dynamic model. In this example, it is shown that the truncation of the series at order 2 yields an accurate analytic approximation which can be used for time simulation and control issues. The relevance of the method is illustrated by simulations.

1. Introduction

This paper addresses the representation of nonlinear systems as a series expansion of linear systems with nonlinear interconnections. It investigates on the well-posedness of such series, the accuracy of truncated sums and their use for application issues.

Such representations were first proposed for finite dimensional systems by Volterra (1959) who introduced the series named after him. There exists a vast literature concerning Volterra series. Among others, they were studied in Brockett (1976), Fliess, Lamnabhi, and Lamnabhi-Lagarrigue (1983), Gilbert (1977) and Isidori (1995) from the geometric control point of view, and in Crouch and Collingwood (1987), Rugh (1981) and Schetzen (1989) from the input–output representation and realization point of view. For linear analytic finite dimensional systems, they correspond to the Taylor series of Frechet derivatives of the input-to-output operator (see Gilbert, 1977 and references therein).

Truncated Volterra series (or their low-order optimized approximations Chen, 2009 and De Figueiredo & Dwyer, 1980) are very convenient for the modeling, identification, model order reduction and real-time simulation of weakly nonlinear systems. This is why they are widely used in signal processing, control, electronics, electromagnetic waves, mechanics, acoustics, bio-medical engineering, etc. However, only a few results about the convergence and truncation error bound are available. The existence of a non zero convergence radius for complex linear analytic finite dimensional systems with zero initial conditions has been proved in Brockett (1977). Other theoretical and local-in-time results are known (see e.g. Isidori, 1995 and Lamnabhi-Lagarrigue, 1994). Results on fading memory have been investigated in Boyd and Chua (1985). More recent results have been obtained in Jing, Lang, and Billings (2008) and Peng and Lang (2007) for the frequency domain, in Bullo (2002) based on regular perturbations, and in Thitsa and Gray (2012) for interconnected systems defined by Fliess series (Fliess et al., 1983). Computable convergence bounds have also been established for finite dimensional linear-analytic systems in Hélie and Laroche (2011).

Here, we address the series representation and the convergence characterization problem for a general class of semi-linear systems, which are analytic in state, affine in input and infinite dimensional. This includes distributed and boundary control systems. We obtain sufficient conditions on both the input and initial condition, under which the series is convergent. Moreover, we give an estimate of

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the error on trajectories, when approximating the original system by the truncated series.

The paper is organized as follows. Section 2 describes the class of systems under consideration and the proposed series expansion. The main results of the paper, that is the convergence and truncation error bounds of the series expansion, are detailed in Section 3 and proved in Section 4. Section 5 points out some additional properties and possible refinements. These results are illustrated on a nonlinear epidemic model in Section 6, for which a simplified approximating model is derived.

2. Systems under consideration

The following notations and functional setting are introduced:

- $\mathbb{T}$ denotes the time interval $[0, T]$ with $T > 0$ or $\mathbb{R}_+$.
- $\mathbb{U}$ and $\mathbb{X}$ are Banach spaces on the field $\mathbb{R}$.
- $\mathcal{L}(\mathbb{U}, \mathbb{X})$ and $\mathcal{L}(\mathbb{X})$ are the sets of bounded linear operators from $\mathbb{U}$ to $\mathbb{X}$, and from $\mathbb{X}$ to $\mathbb{X}$, respectively.
- $\mathcal{ML}_k(\mathbb{X}, \mathbb{X})$ $(k \geq 2)$ is the set of bounded multilinear operators from $\underbrace{\mathbb{X} \times \cdots \times \mathbb{X}}_{k}$ to $\mathbb{X}$, with norm $\|E\| = \sup_{\substack{(x_1,\ldots,x_k)\in\mathbb{X}^k \\ \|x_1\|=\cdots=\|x_k\|=1}} \|E(x_1, \ldots, x_k)\|_{\mathbb{X}}$.
- $\mathcal{ML}_{k,1}(\mathbb{X}, \mathbb{U}, \mathbb{X})$ $(k \geq 1)$ is the set of bounded multilinear operators from $\underbrace{\mathbb{X} \times \cdots \times \mathbb{X}}_{k} \times \mathbb{U}$ to $\mathbb{X}$, with norm $\|E\| = \sup_{\substack{(x_1,\ldots,x_k,u)\in\mathbb{X}^k\times\mathbb{U} \\ \|x_1\|=\cdots=\|x_k\|=\|u\|=1}} \|E(x_1, \ldots, x_k, u)\|_{\mathbb{X}}$.
- $\mathcal{U} = L^\infty(\mathbb{T}, \mathbb{U})$ and $\mathcal{X} = L^\infty(\mathbb{T}, \mathbb{X})$ are standard Lebesgue spaces.

We consider the class of infinite-dimensional nonlinear control systems on $\mathbb{X}$, having an equilibrium state (shifted to zero without loss of generality), governed by

$$\dot{x} = L(x, u) + P(x) + Q(x, u), \quad \text{for } t \in \mathbb{T}, \tag{1}$$

$$x(0) = x_{\text{ini}} \in \mathbb{X}. \tag{2}$$

The notation $L(x, u)$ stands for the linear part of the system. We assume that the linearized system

$$\dot{x}_1 = L(x_1, u), \qquad x_1(0) = x_{\text{ini}} \in \mathbb{X}, \tag{3}$$

is a distributed or boundary control system in the sense of Curtain and Zwart (1995). This implies that $A = L(., 0)$ generates a strongly continuous semigroup on $\mathbb{X}$, denoted V, with $\alpha \in \mathbb{R}$ and $\beta > 0$ such that for all $t \in \mathbb{T}$, $\|V(t)\| \leq \beta e^{\alpha t}$. The growth bound α is assumed to be strictly negative if $\mathbb{T} = \mathbb{R}_+$.

Moreover, the linearized system (3) is assumed to be well-posed, that is, for all $u \in \mathcal{U}$ and $x_{\text{ini}} \in \mathbb{X}$, (3) has a unique mild solution $x_1 \in \mathcal{X}$. P and Q are nonlinear terms such that

$$P(x) = \sum_{k=2}^{+\infty} A_k(\underbrace{x, \ldots, x}_{k}), \tag{4}$$

$$Q(x, u) = \sum_{k=2}^{+\infty} B_k(\underbrace{x, \ldots, x}_{k-1}, u), \tag{5}$$

where $A_k \in \mathcal{ML}_k(\mathbb{X}, \mathbb{X})$ and $B_k \in \mathcal{ML}_{k-1,1}\big(\mathbb{X}, \mathbb{U}, \mathbb{X}\big)$ are multilinear bounded operators. The complex functions

$$a : z \longmapsto \sum_{k=2}^{+\infty} \|A_k\| \, z^k, \qquad b : z \longmapsto \sum_{k=2}^{+\infty} \|B_k\| \, z^k, \tag{6}$$

are assumed to be analytic at $z = 0$.

A series expansion of the trajectories of (1)–(2) is defined in the following way. For all $m \geq 2$, x_m is the mild solution of

$$\dot{x}_m = Ax_m + \chi_m, \qquad x_m(0) = 0, \tag{7}$$

where

$$\chi_m(\tau) = \sum_{k=2}^{m} \sum_{p\in\mathbb{M}_m^k} A_k\Big(x_{p_1}(\tau), \ldots, x_{p_k}(\tau)\Big) + \sum_{k=2}^{m} \sum_{\begin{cases} q \in \mathbb{M}_m^k \\ q_k = 1 \end{cases}} B_k\Big(x_{q_1}(\tau), \ldots, x_{q_{k-1}}(\tau), u(\tau)\Big), \tag{8}$$

where $\mathbb{M}_m^K$ is defined for all $m \in \mathbb{N}^*$ and $K \in \mathbb{N}^*$ by

$$\mathbb{M}_m^K = \Big\{p \in (\mathbb{N}^*)^K \mid p_1 + \cdots + p_K = m\Big\}.$$

As a well known result (Curtain & Zwart, 1995; Pazy, 1983), we have

$$x_m(t) = \int_0^t V(t - \tau)\chi_m(\tau) \, \mathrm{d}\tau. \tag{9}$$

The series expansion of the trajectories is

$$x(t) = \sum_{m=0}^{+\infty} x_m(t). \tag{10}$$

It provides a formal solution of (1)–(2).

In the case of finite dimensional linear analytic systems with zero initial conditions, the semigroup associated with the linearized system is $V(t) = e^{At}$, the solution x_1 is the convolution of the input by the impulse response matrix VB. Moreover, (9) corresponds to a multiple convolution (of order m) by a multivariate kernel and (10) exactly coincides with a standard Volterra series expansion (see e.g. Boyd, Chua, & Desoer, 1984; Rugh, 1981; Volterra, 1959). It is shown in Gilbert (1977) that for such systems, this expansion is indeed the Taylor series of Frechet derivatives of the input-to-state operator.

A realization of the partial sum of order three of (10) is displayed in Fig. 1. Each term is built as a cascade of linear systems (*lin*, V) and static nonlinear interconnections (A_k, B_k), which provides an easily implementable simplified model. Moreover, this realization is directly expressed in terms of the original system parameters, which constitutes an appealing feature for design issues and physical interpretations.

From a general point of view, approximations by low-order truncated Taylor series are well-adapted to "weakly nonlinear systems for sufficiently small inputs": Section 3 provides quantitative assessment criteria for this statement, based on the system parameters. More precisely, we establish (i) a guaranteed convergence domain of (10) with respect to the input and the initial conditions, and (ii) an estimate of the remainder with respect to the truncation order.

For applications requiring a low-order approximation even for large inputs, optimal approximations may be preferred to truncation. Although this is beyond the scope of this paper, results (i–ii) can still be helpful in this case. Indeed, result (i) provides a range over which optimal approximations defined in e.g. Chen (2009) and De Figueiredo and Dwyer (1980) are guaranteed to make sense. Moreover, result (ii) provides a guaranteed estimate to which the optimal approximation error can be compared.

3. Main results

Our first main result is a sufficient condition on x_{ini} and u for the convergence of the series (10), for which we need to introduce the following definitions.

For all $t \in \mathbb{T}$, we set

$$f(t) = \max\left[\sup_{\begin{cases} k \geq 2 \text{ s.t.} \\ \|A_k\| \neq 0 \end{cases}} \frac{\|V(t)\, A_k\|}{\|A_k\|}, \ \sup_{\begin{cases} k \geq 2 \text{ s.t.} \\ \|B_k\| \neq 0 \end{cases}} \frac{\|V(t)\, B_k\|}{\|B_k\|}\right].$$