Advances in Engineering Software 122 (2018) 13–21

Contents lists available at ScienceDirect

Advances in Engineering Software

journal homepage: www.elsevier.com/locate/advengsoft

Research paper

An enhanced explicit technique for the solution of non-Fourier heat transfer problems

Delfim Soares Jr.

Structural Engineering Department, Federal University of Juiz de Fora, Juiz de Fora, MG, Brazil

ARTICLE INFO

Keywords:
Hyperbolic heat conduction
Non-Fourier models
Time-marching techniques
Explicit Analysis
Stability
Numerical dissipation

ABSTRACT

In this work, an enhanced explicit technique is proposed to analyze hyperbolic heat conduction models. As usual, the explicit approach allows the solution of the problem to be carried out without dealing with any system of equations, featuring a very efficient methodology. In addition, the proposed technique enables algorithmic dissipation, allowing the influence of spurious high modes to be properly eliminated, without introducing significant period elongation and amplitude decay errors into the analysis. As an explicit approach, the technique is conditionally stable; however, it exhibits high stability limits (its critical time-step is around 1.8 times that of the Central Difference Method), emphasizing its effectiveness. The technique is very accurate, truly self-starting and extremely direct to implement. At the end of the manuscript, numerical results are presented, illustrating the good performance of the discussed technique.

1. Introduction

Non-Fourier models have been widely explored in the last years, since they provide a plausible physical explanation for the behavior of heat transfer in models where the standard Fourier conduction law does not properly apply. This has been the case, for instance, considering heat transfer in cryogenic fluid and biomaterials, as well as considering the micro- and nano-scales ultrafast transient heating of metals.

The main difference between the traditional Fourier heat conduction model and the non-Fourier hyperbolic formulation is that the later takes into account the wave nature of thermal transfer. As described by Wu et al. [1], many works have been published in the solution and analysis of non-Fourier models, in the last decades: Carey and Tsai [2], for instance, presented some numerical methods for 1D cases, combining the Finite Element Method for the spatial domain and two difference schemes for the time domain; Manzari et al. [3,4] presented a numerical approach, combining the Galerkin method and the Crank-Nicolson method; Zhang et al. [5,6] suggested a approach for 1D heat equations, employing a fourth-order boundary value method for discretizing the temporal variable and a fourth-order compact difference scheme for discretizing the spatial variable; Monteiro et al. [7] developed a method for the hyperbolic heat transfer in a finite slab, using a generalized integral transform and the Gear method; Saleh and Al-Nimr [8] developed a variational formulation applying the Laplace transform technique to overcome the oscillations of the numerical results; Chen et al. [9–12] and Loureiro et al. [13] developed methods based on Green functions for the hyperbolic heat equation, and discussed 1D and 2D cases; Hsu [14] employed the differential quadrature method; Roy et al. [15] suggested a new difference scheme based on the multiple scale technique; Miller and Haber [16] presented a new space-time discontinuous Galerkin FEM for the hyperbolic heat conduction problem; Movahedian and Boroomand [17] proposed a solution method using exponential basis functions; and Han [18,19] discussed the finite volume solution of 1D and 2D problems, considering homogeneous and heterogeneous media.

In the present work, a new explicit formulation is proposed to analyze non-Fourier hyperbolic heat conduction models. Explicit procedures are usually preferable because of their lower computational effort, allowing responses to be obtained without the necessity to deal with the solution of any system of equations; however, there are restrictions in their use due to stability conditions. The procedure that is discussed here is an extension of the previous works of Soares [20,21], which were mainly concerned with dynamic analyzes. In the present methodology, algorithmic dissipation is enabled, allowing overcoming spurious numerical oscillations. The technique is very simple and direct to implement, and it is only based on single-step relations of the temperature and its first time derivative, characterizing a truly self-starting procedure. In addition, the formulation exhibits enhanced accuracy and it has extended stability limits, with its critical time-step being around 1.8 times that of the classical Central Difference Method (CDM). Thus, it minimizes the main drawback of explicit procedures, allowing time-steps that are usually adopted in implicit analyzes to be considered, enabling good results to be calculated at much reduced computational costs.

E-mail address: delfim.soares@ufjf.edu.br.

https://doi.org/10.1016/j.advengsoft.2018.02.010
Received 5 December 2017; Received in revised form 9 January 2018; Accepted 4 February 2018
0965-9978/ © 2018 Elsevier Ltd. All rights reserved.

The manuscript is organized as follows: first, the governing equations are briefly presented, describing the problem to be analyzed once a spatial discretization technique is applied to the hyperbolic heat conduction model; in the sequence, the proposed explicit time marching technique is described, and some of its properties are discussed; finally, numerical examples are presented, illustrating the good performance of the proposed methodology.

2. Governing equations

The classical Fourier conduction law yields temperature solutions which imply in an infinite speed of heat propagation. In order to eliminate this anomaly, Cattaneo[22] and Vernotte [23] formulated a time-dependent relaxation model for the heat flux in solids, introducing a relaxation time τ into the analysis. Thus, the heat flux vector $\mathbf{q}$ became related to the temperature gradient ∇T as:

$$\mathbf{q}(\mathbf{x}, t) + \tau\, \dot{\mathbf{q}}(\mathbf{x}, t) = -k\nabla T(\mathbf{x}, t) \tag{1}$$

where over dots indicate time derivatives and k stands for the thermal conductivity. If $\tau = 0$ is considered, Eq. (1), which describes the non-Fourier model in focus, reduces to the classical Fourier conduction law.

By associating Eq. (1) to the energy conservation law, the governing equation of the hyperbolic heat conduction can be deduced, and it may be expressed as:

$$\tau\rho c\ddot{T}(\mathbf{x}, t) + \rho c\dot{T}(\mathbf{x}, t) - \nabla\cdot k\nabla T(\mathbf{x}, t) = Q(\mathbf{x}, t) \tag{2}$$

where ρ and c stand for the mass density and specific heat capacity, respectively, and Q stands for source terms.

Once a spatial discretization technique is employed (as for instance, the Finite Element Method [24,25]) taking into account the governing Eq. (2), Eq. (3) may be obtained:

$$\tau\mathbf{C}\ddot{\mathbf{T}}(t) + \mathbf{C}\dot{\mathbf{T}}(t) + \mathbf{K}\mathbf{T}(t) = \mathbf{F}(t) \tag{3}$$

where $\mathbf{K}$ stands for the thermal conductance matrix, $\mathbf{C}$ represents the heat capacity matrix, $\mathbf{F}$ is the heat load vector and $\mathbf{T}$ is the temperature vector.

In the next section, the time domain solution of Eq. (3) is discussed, taking into account an enhanced explicit procedure. As one will observe, once lumped heat capacity matrices are considered, the proposed algorithm becomes very efficient, generating no system of equations to be dealt with at each time step of the analysis. In addition, as described by previous authors [1], lumped heat capacity matrices may also exhibit further advantages, such as better resistance to numerical oscillations. Thus, lumped $\mathbf{C}$ is considered here.

3. Time-domain solution

Considering a regular time discretization, which is defined by a time-step Δt (i.e., $t^{n+1} = t^n + \Delta t$), the following algorithm enables the explicit solution of Eq. (3):

$$\dot{\mathbf{T}}^{n+1} = a_0\mathbf{C}^{-1}(\mathbf{F}^{n+1} + \mathbf{F}^n) + a_1\dot{\mathbf{T}}^n - \mathbf{C}^{-1}\mathbf{K}(a_2\mathbf{T}^n + a_3\dot{\mathbf{T}}^n) \tag{4a}$$

$$\mathbf{T}^{n+1} = \mathbf{T}^n + a_4\dot{\mathbf{T}}^n + a_5\dot{\mathbf{T}}^{n+1} - \mathbf{C}^{-1}\mathbf{K}(a_6\dot{\mathbf{T}}^n + a_7\dot{\mathbf{T}}^{n+1}) \tag{4b}$$

where the integration constants a are defined as: $a_0 = \Delta t/(2\tau + \Delta t)$; $a_1 = (2\tau - \Delta t)/(2\tau + \Delta t)$; $a_2 = 2a_0$; $a_3 = \Delta t a_0$; $a_4 = \Delta t/2$; $a_5 = a_4 - a_3$; $a_6 = 0.007359\Delta t^2 a_2$; and $a_7 = 0.071067\Delta t^2 a_2$.

As one can observe, Eqs. (4a,b) stand as a very simple truly self-starting explicit approach and their implementation is very direct. They require no system of equations to be dealt with (as previously remarked, diagonal $\mathbf{C}$ matrices are considered), as well as they do not deal with higher time derivatives of the temperature field and/or field values at multiple time steps; i.e., just the temperature field itself and its first time derivative are considered in Eqs. (4a,b), taking into account their values in the current and previous time steps of the time marching process.

It is important to highlight that Eqs. (4a,b) are mathematically constructed; i.e., the above expressions and constants were mathematically established so that a group of positive features would be introduced into the solution process. Thus, equations (4) are not based on simple approximate assumptions, but on connected mathematical efforts. Basically, Eqs. (4a,b) were developed so that the method would exhibit large critical stability limits and it would provide elevated algorithmic dissipation in the higher modes of the problem. The algorithm described by Eqs. (4a,b) can be stated as an evolution and generalization of the method proposed by Soares [20], for dynamic analysis. In this context, Eq. (4a) and the initial terms of Eq. (4b) were adapted from [20], following an explicit pattern, and the term $-a_3\dot{\mathbf{T}}^{n+1} - \mathbf{C}^{-1}\mathbf{K}(a_6\dot{\mathbf{T}}^n + a_7\dot{\mathbf{T}}^{n+1})$ was added to the methodology (Eq. (4b)), providing spectral radius curves with a single bifurcation point that occurs at large sampling frequencies and that relates to low spectral radius values (in fact, the methodology was formulated so that the spectral radius of the method would become null at the bifurcation point, for $\tau \to \infty$). In addition, the algorithm was developed so that a unique critical stability limit would occur.

As usual, in order to better analyze the properties of the proposed technique, the following single-degree-of-freedom (SDOF) model can be studied:

$$\ddot{T}(t) + \varsigma\,\dot{T}(t) + \omega^2 T(t) = f(t) \tag{5}$$

where $\varsigma = \tau^{-1}$ and ω^2 relates to an eigenvalue of the model, which is computed based on matrices $\mathbf{K}$ and $\tau\mathbf{C}$.

Considering Eq. (5) and the solution algorithm (4), the following recursive relationship may be obtained:

$$\begin{bmatrix} T^{n+1} \\ \dot{T}^{n+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} T^n \\ \dot{T}^n \end{bmatrix} + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} f^n \\ f^{n+1} \end{bmatrix} = \mathbf{A}\begin{bmatrix} T^n \\ \dot{T}^n \end{bmatrix} + \mathbf{L}\begin{bmatrix} f^n \\ f^{n+1} \end{bmatrix} \tag{6}$$

where $\mathbf{A}$ and $\mathbf{L}$ stand for the amplification and the load operator matrices, respectively, and the entries of $\mathbf{A}$ are given by (rounded values are considered):

$$A_{11} = (4 + 4\varsigma\Delta t + \varsigma^2\Delta t^2 - 2\omega^2\Delta t^2 + \varsigma\omega^2\Delta t^3 + 0.284268\omega^4\Delta t^4)/(2 + \varsigma\Delta t)^2 \tag{7a}$$

$$A_{12} = (4 + \varsigma^2\Delta t^2 - 1.313704\omega^2\Delta t^2 + 0.627416\varsigma\omega^2\Delta t^3 + 0.142134\omega^4\Delta t^4)\Delta t/(2 + \varsigma\Delta t)^2 \tag{7b}$$

$$A_{21} = -2w^2\Delta t/(2 + \varsigma\Delta t) \tag{7c}$$

$$A_{22} = (2 - \varsigma\Delta t - \omega^2\Delta t^2)/(2 + \varsigma\Delta t) \tag{7d}$$

The stability condition requires that matrix $\mathbf{A}$ does not amplify errors as the time-step algorithm advances on time. The conditions required to assure stability are [24]: (i) $\rho(\mathbf{A}) \leq 1$; (ii) eigenvalues of $\mathbf{A}$ of multiplicity greater than one are strictly less than one in modulus. In item (i), $\rho(\mathbf{A})$ is the spectral radius of matrix $\mathbf{A}$, which represents the maximal absolute magnitude of the eigenvalues of $\mathbf{A}$.

The eigenvalues of the amplification matrix are given by:

$$\lambda_{1,2}(\mathbf{A}) = A_1 \pm (A_1^2 - A_2)^{1/2} \tag{8}$$

where A_1 is half the trace of matrix $\mathbf{A}$ and A_2 is the determinant of $\mathbf{A}$. Taking into account the proposed methodology, A_1 and A_2 may be expressed as:

$$A_1 = (4 + 2\varsigma\Delta t - 2\omega^2\Delta t^2 + 0.142134\omega^4\Delta t^4)/(2 + \varsigma\Delta t)^2 \tag{9a}$$

$$A_2 = (4 - \varsigma^2\Delta t^2 - 0.029436\omega^4\Delta t^4)/(2 + \varsigma\Delta t)^2 \tag{9b}$$

which allow establishing the stability limit of the method.

In this way, the critical sampling frequency of the model Ω_c, which is the value of $\Omega = \omega\Delta t$ under which stability is ensured, is given by $\Omega_c = 3.570835$, considering the present technique. It is important to observe that this value does not depend on ς; in fact, Eqs. (4a,b) are

Download English Version:

https://daneshyari.com/en/article/6961280

Download Persian Version:

https://daneshyari.com/article/6961280

Daneshyari.com