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High-order three-scale computational method for heat conduction problems of axisymmetric composite structures with multiple spatial scales

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ABSTRACT

This study develops a novel high-order three-scale (HOTS) computational method for heat conduction problems of axisymmetric composite structures with multiple spatial scales. The heterogeneities of the composites are taken into account by periodic distributions of unit cells on the mesoscale and microscale. Firstly, the multiscale asymptotic analysis for these multiscale problems is given detailedly. Based on the above-mentioned analysis, the new unified micro-meso-macro HOTS approximate solutions are successfully constructed for these multiscale problems. Two classes of auxiliary cell functions are established on the mesoscale and microscale. Then, the error analyses for the conventional two-scale solutions, low-order three-scale (LOTS) solutions and HOTS solutions are obtained in the pointwise sense, which illustrate the necessity of developing HOTS solutions for simulating the heat conduction behaviors of composite structures with multiple periodic configurations. Furthermore, the corresponding HOTS numerical algorithm based on finite element method (FEM) is brought forward in details. Finally, some numerical examples are presented to verify the feasibility and validity of our HOTS computational method. In this paper, a unified three-scale computational framework is established for heat conduction problems of axisymmetric composite structures with multiple spatial scales.

1. Introduction

With the rapid development of modern material science, the composite materials have been widely applied to aeronautic and aerospace engineering owing to their excellent thermal properties. These composites are usually served under complex thermal environments. Hence, the accurate evaluation of the heat transferring behaviors of these composites has attracted lots of attentions from scientists and engineers. Generally, the equation, which governs the heat conduction process for these composites, has rapidly oscillatory coefficients due to the sharply varying between different components of the composites. To our knowledge, the direct numerical computation for these multiscale problems needs a tremendous amount of computational resources to capture microscale behaviors [1–4]. In the past few decades, mathematicians and engineers have developed some multiscale methods to study the multiscale behaviors of the composite materials, such as the asymptotic homogenization method (AHM), heterogeneous multiscale method (HMM), variational multiscale method (VMS), and multiscale finite element method (MsFEM) [5–8], etc. Among them, the AHM is widely used because it has a rigorous mathematical foundation and can combine with FEM very well. In consideration of the low numerical accuracy of conventional homogenization method, Cui et al. systematically developed a second-order two-scale (SOTS) analysis method to accurately predict the physical and mechanical behaviors of composites, which provides a feasible framework of two-scale computation for the composite materials [9–15].

In recent years, the composite materials with several spatial scales have been designed and manufactured for engineering application. To the best of our knowledge, some studies have been performed on the composite materials with several spatial scales. But they only discussed the composites with periodic structures in cartesian coordinates [16–27]. In addition, most of them simply concentrate on computing and predicting the effective material parameters of these composite materials [17–20,23–27]. Sometimes, these composites are designed as axisymmetric structures for practical application such as cylindrical pressure vessel, culvert pipe, tube of the artillery, etc. Up to now, some researches have been performed on the heat conduction problems of the composites with axisymmetric structures [11,28–30]. However, their researches only focused on the composites with two spatial scales. Unfortunately, their two-scale computational methods can not be used

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to solve heat conduction problems of axisymmetric composite structures with multiple spatial scales. The reason is that we should need a tremendous amount of computational resources to solve the auxiliary cell problems when we apply the two-scale methods to the composite materials with multiple spatial scales. Moreover, the two-scale methods can not capture the enough oscillating information of the minimum scale of the composites. All in all, there is a lack of adequate researches on heat conduction problems of axisymmetric composite structures with multiple spatial scales.

The aim of this paper is to develop a HOTS computational method for heat conduction problems of axisymmetric composite structures with multiple spatial scales. To accurately analyze the heat conduction problems of axisymmetric composite structures with multiple spatial scales, we construct the HOTS solutions for these multiscale problems in virtue of the SOTS method [9–12,31] and reiterated homogenization method [18–20]. Then we propose a corresponding multiscale numerical algorithm based on FEM to efficiently solve these multiscale problems. Furthermore, it should state that our high-order solutions contain the essential high-order correctors compared to classical three-scale solutions in [17–20], which can help us precisely capture the microscale oscillation information and significantly improve the numerical accuracy of multiscale solutions for these multiscale problems.

This paper is outlined as follows. In Section 2, the detailed construction of the HOTS solutions for heat conduction problems of axisymmetric composite structures with multiple spatial scales is given by multiscale asymptotic analysis. Moreover, the error analysis in the pointwise sense of HOTS solutions is presented. By comparing the results of error analysis of conventional two-scale solutions, the low-order three-scale (LOTS) solutions and HOTS solutions, we theoretically explain the importance of HOTS solutions in capturing microscale oscillation information. In Section 3, a HOTS numerical algorithm based on FEM is presented to effectively and accurately solve these multiscale problems. In Section 4, some numerical results are shown to verify the validity of our HOTS algorithm. Finally, some useful conclusions are given in Section 5.

For convenience, throughout this paper we use the Einstein summation convention on repeated indices.

2. The multiscale asymptotic analysis

Let us consider the governing equation for heat conduction problems of axisymmetric composite structures with three spatial scales as follows

$$\begin{cases} \dfrac{\partial q_r^{\varepsilon_1\varepsilon_2}(\mathbf{x})}{\partial r} + \dfrac{\partial q_z^{\varepsilon_1\varepsilon_2}(\mathbf{x})}{\partial z} + \dfrac{q_r^{\varepsilon_1\varepsilon_2}(\mathbf{x})}{r} = h(\mathbf{x}), & \text{in } \Omega,\\ T^{\varepsilon_1\varepsilon_2}(\mathbf{x}) = \widehat{T}(\mathbf{x}), & \text{on } \partial\Omega_T,\\ -q_i^{\varepsilon_1\varepsilon_2}(\mathbf{x})n_i = \bar{q}(\mathbf{x}), & \text{on } \partial\Omega_q. \end{cases} \tag{1}$$

where Ω is a bounded convex domain in $\mathbb{R}^2$ with a boundary $\partial\Omega = \partial\Omega_T \cup \partial\Omega_q$. In the mesoscale, the domain Ω can be regarded as a set of mesoscopic cell Y, and then in the microscale, mesoscopic cell Y can be also regarded as a set of microscopic cell Z, as shown in Fig. 1. The sizes of mesoscopic cell Y and microscopic cell Z are ε_1 and ε_2, respectively, and we suppose that the small parameter ε_2 has the same order of magnitude with ε_1^2. Besides, $\widehat{T}(\mathbf{x})$ is the prescribed temperature on the boundary $\partial\Omega_T$, and $\bar{q}(\mathbf{x})$ is the heat flux prescribed normal to the boundary $\partial\Omega_q$ with the normal vector n_i; h is the internal heat source; The $T^{\varepsilon_1\varepsilon_2}$ in (1) is the undetermined temperature field which contains in the heat flux $q_i^{\varepsilon_1\varepsilon_2}$. Now, let us set macro-coordinates $\mathbf{x} = (x_1, x_2) = (r, z)$, meso-coordinates $\mathbf{y} = \dfrac{\mathbf{x}}{\varepsilon_1} = (y_1, y_2)$ and micro-coordinates $\mathbf{z} = \dfrac{\varepsilon_1\mathbf{y}}{\varepsilon_2} = (z_1, z_2)$. Then the governing equation (1) can be rewritten as follows

$$\begin{cases} \dfrac{\partial q_i^{\varepsilon_1\varepsilon_2}(\mathbf{x})}{\partial x_i} + \dfrac{q_1^{\varepsilon_1\varepsilon_2}(\mathbf{x})}{x_1} = h(\mathbf{x}), & \text{in } \Omega,\\ T^{\varepsilon_1\varepsilon_2}(\mathbf{x}) = \widehat{T}(\mathbf{x}), & \text{on } \partial\Omega_T,\\ q_i^{\varepsilon_1\varepsilon_2}(\mathbf{x})n_i = \bar{q}(\mathbf{x}), & \text{on } \partial\Omega_q. \end{cases} \tag{2}$$

In this paper, we define the constitutive law of multiscale problem (2) as follows

$$q_i^{\varepsilon_1\varepsilon_2}(\mathbf{x}) = -k_{ij}^{\varepsilon_1\varepsilon_2}(\mathbf{x})\frac{\partial T^{\varepsilon_1\varepsilon_2}(\mathbf{x})}{\partial x_j}. \tag{3}$$

Due to ε_1 periodicity of structure Ω in the mesoscale and ε_2 periodicity of cell Y in the microscale, material parameters $k_{ij}^{\varepsilon_1\varepsilon_2}$ can be expressed as the following different forms

$$k_{ij}^{\varepsilon_1\varepsilon_2}(\mathbf{x}) = k_{ij}^{\varepsilon_2}(\mathbf{y}) = k_{ij}(\mathbf{y}, \mathbf{z}) \tag{4}$$

Then, according to the definitions of macro-coordinates, meso-coordinates and micro-coordinates, there exists the chain rules on multivariable derivatives as follows

$$\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon_1}\frac{\partial}{\partial y_i},\quad \frac{\partial}{\partial y_i} \to \frac{\partial}{\partial y_i} + \frac{\varepsilon_1}{\varepsilon_2}\frac{\partial}{\partial z_i} \tag{5}$$

And then

$$\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon_1}\frac{\partial}{\partial y_i} + \frac{1}{\varepsilon_2}\frac{\partial}{\partial z_i} \tag{6}$$

which will be extensively used in the sequel.

2.1. The high-order three-scale analysis of governing equation

In this subsection, HOTS asymptotic solutions are derived for evaluating the heat conduction behaviors of axisymmetric composite structures with three spatial scales. To the problem (2), we suppose that $T^{\varepsilon_1\varepsilon_2}(\mathbf{x})$ can be expanded as the following asymptotic expansion form

$$\begin{aligned} T^{\varepsilon_1\varepsilon_2}(\mathbf{x}) = {} & T^{(0)}(\mathbf{x},\mathbf{y},\mathbf{z}) + \varepsilon_1 T^{(1)}(\mathbf{x},\mathbf{y},\mathbf{z}) + \varepsilon_2 T^{(2)}(\mathbf{x},\mathbf{y},\mathbf{z})\\ & + \varepsilon_1^2 T^{(3)}(\mathbf{x},\mathbf{y},\mathbf{z}) + \varepsilon_1\varepsilon_2 T^{(4)}(\mathbf{x},\mathbf{y},\mathbf{z}) + \varepsilon_2^2 T^{(5)}(\mathbf{x},\mathbf{y},\mathbf{z})\\ & + \varepsilon_1^2\varepsilon_2 T^{(6)}(\mathbf{x},\mathbf{y},\mathbf{z}) + \varepsilon_1^{-2}\varepsilon_2^3 T^{(7)}(\mathbf{x},\mathbf{y},\mathbf{z}) + \varepsilon_1\varepsilon_2^2 T^{(8)}(\mathbf{x},\mathbf{y},\mathbf{z})\\ & + \varepsilon_1^{-1}\varepsilon_2^3 T^{(9)}(\mathbf{x},\mathbf{y},\mathbf{z}) + \varepsilon_1^{-3}\varepsilon_2^4 T^{(10)}(\mathbf{x},\mathbf{y},\mathbf{z}) + o(\varepsilon_2^2). \end{aligned} \tag{7}$$

In order to simplify the analysis process, we firstly define the following operators

$$\begin{aligned} L_0(T) &= -\frac{\partial}{\partial x_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial x_j}\right]\\ L_1(T) &= -\frac{\partial}{\partial x_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial y_j}\right] - \frac{\partial}{\partial y_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial x_j}\right]\\ L_2(T) &= -\frac{\partial}{\partial x_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial z_j}\right] - \frac{\partial}{\partial z_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial x_j}\right]\\ L_3(T) &= -\frac{\partial}{\partial y_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial y_j}\right]\\ L_4(T) &= -\frac{\partial}{\partial y_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial z_j}\right] - \frac{\partial}{\partial z_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial y_j}\right]\\ L_5(T) &= -\frac{\partial}{\partial z_i}\left[k_{ij}(\mathbf{y},\mathbf{z})\frac{\partial T}{\partial z_j}\right] \end{aligned} \tag{8}$$

Now substituting (8) into the heat conduction equation of original problem (2), expanding the derivatives and matching terms with the same order of small periodic parameters ε_1 and ε_2, we can immediately obtain the following equation