Automatica 49 (2013) 2939–2952

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Discrete-time stochastic control systems: A continuous Lyapunov function implies robustness to strictly causal perturbations[☆]

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ARTICLE INFO

Article history:
Received 13 January 2013
Received in revised form
31 May 2013
Accepted 10 June 2013
Available online 7 August 2013

Keywords:
Stochastic systems
Lyapunov stability
Robust stability

ABSTRACT

Discrete-time stochastic systems employing possibly discontinuous state-feedback control laws are addressed. Allowing discontinuous feedbacks is fundamental for stochastic systems regulated, for instance, by optimization-based control laws. We introduce generalized random solutions for discontinuous stochastic systems to guarantee the existence of solutions and to generate enough solutions to get an accurate picture of robustness with respect to strictly causal perturbations. Under basic regularity conditions, the existence of a continuous stochastic Lyapunov function is sufficient to establish that asymptotic stability in probability for the closed-loop system is robust to sufficiently small, state-dependent, strictly causal, worst-case perturbations. Robustness of a weaker stochastic stability property called recurrence is also shown in a global sense in the case of state-dependent perturbations, and in a semiglobal practical sense in the case of persistent perturbations. An example shows that a continuous stochastic Lyapunov function is not sufficient for robustness to arbitrarily small worst-case disturbances that are not strictly causal. Our positive results are also illustrated by examples.

1. Introduction

It is known that there exist stabilizable deterministic discrete-time nonlinear control systems that cannot be stabilized by continuous state feedback (Rawlings & Mayne, 2009, Example 2.7) even though they admit a continuous control-Lyapunov function (Grimm, Messina, Tuna, & Teel, 2005, Example 1) and thus can be *robustly* stabilized by *discontinuous* state feedback (Kellett & Teel, 2004). For instance, the class of Model Predictive Control (MPC) feedback laws does allow discontinuous stabilizing control laws (Grimm et al., 2005; Messina, Tuna, & Teel, 2005; Rawlings & Mayne, 2009). Since the MPC feedback law may be discontinuous, having a continuous Lyapunov function for the closed-loop system is necessary to establish nominal robustness (Grimm et al., 2005; Kellett & Teel, 2004). On the other hand, there exist discrete-time systems stabilized by discontinuous control laws, but absolutely non-robust due to the lack of continuous Lyapunov functions (Grimm, Messina, Tuna, & Teel, 2004).

At least to the authors' knowledge, there are no similar robustness results for the class of *stochastic* systems under discontinuous control laws. This fact motivates our investigations.

Regarding stochastic systems, different stability notions and Lyapunov conditions have been studied in the literature (Kolmanovskii & Shaikhet, 2002; Kozin, 1969; Kushner, 1967, 1971; Meyn, 1989; Meyn & Tweedie, 1993). Recently, there has been interest regarding stochastic systems with non-unique solutions (Teel, 2009) due to the interaction between random inputs and worst-case behavior. In Teel (in press) the notion of random solutions to set-valued discrete-time stochastic systems is introduced. The equivalence between the existence of a continuous Lyapunov function and global asymptotic stability in probability of a compact attractor for stochastic difference inclusions without control inputs is established in Teel, Hespanha, and Subbaraman (submitted for publication) under certain regularity assumptions. A similar result showing the equivalence between the existence of a smooth Lyapunov function and a weaker stochastic stability property called recurrence is presented in Subbaraman and Teel (2013).

In this paper, we consider discrete-time stochastic systems with basic regularity properties and we investigate robustness of asymptotic stability in probability and of recurrence. Since we deal with discontinuous systems, we introduce generalized random solutions to generate enough random solutions which provide an

☆ Research supported in part by the National Science Foundation grant number NSF ECCS-1232035 and the Air Force Office of Scientific Research grant number AFOSR FA9550-12-1-0127. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Valery Ugrinovskii under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2013.06.021

accurate picture of robustness with respect to strictly causal perturbations.

We establish that under the existence of a locally bounded, possibly discontinuous control law that guarantees the existence of a continuous stochastic Lyapunov function for the closed-loop system, asymptotic stability in probability of the attractor is robust to sufficiently small, state-dependent, strictly causal, worst-case perturbations. An example shows that without strict causality we may have no robustness even to arbitrarily small perturbations.

We also show that recurrence of open neighborhoods of the attractor is robust to such kind of sufficiently small perturbations, both state-dependent and persistent perturbations, and in the latter case the robustness that we establish is semiglobal practical robustness.

The set-valued mappings studied here satisfy the basic regularity properties considered in Teel et al. (submitted for publication). This further allows us to also relate the existence of a continuous stochastic Lyapunov function for the nominal closed-loop system to certain stochastic stability properties of the perturbed closed-loop system, in view of the results in Teel et al. (submitted for publication).

The paper is organized as follows. Section 2 contains the basic notation and definitions. In Section 3 we present the class of discrete-time stochastic systems along with certain regularity and Lyapunov conditions. The main results are shown in Section 4. Section 5 introduces the notion of generalized random solutions. Our results are related to stochastic stability properties respectively in Sections 6 and 7. An illustrative MPC example is provided in Section 8. Concluding comments are presented in Section 9. All the proofs are given in the appendices for ease of presentation.

2. Notation

We adopt the notation of Teel et al. (submitted for publication). $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) denotes the set of non-negative (positive) real numbers, and $\mathbb{Z}_{\geq 0}$ ($\mathbb{Z}_{>0}$) denotes the set of non-negative (positive) integers. For any closed set $\mathcal{C} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_{\mathcal{C}} := \inf_{y\in\mathcal{C}} |x-y|$ is the Euclidean distance to the set $\mathcal{C}$. $\mathbb{B}$ ($\mathbb{B}^\circ$) denotes the closed (open) unit ball in $\mathbb{R}^n$. For any set $\mathcal{S} \subseteq \mathbb{R}^n$, the notation $\mathrm{cl}(\mathcal{S})$ denotes the closure of $\mathcal{S}$. For any closed set $\mathcal{C}$ and $\varepsilon \in \mathbb{R}_{>0}$, $\mathcal{C}+\varepsilon\mathbb{B}$ denotes the set $\{x \in \mathbb{R}^n \mid |x|_{\mathcal{C}} \leq \varepsilon\}$. For any set $\mathcal{S} \subseteq \mathbb{R}^n$, we define the indicator function $\mathbb{I}_{\mathcal{S}} : \mathbb{R}^n \to \{0, 1\}$ as $\mathbb{I}_{\mathcal{S}}(x) = 1$ if $x \in \mathcal{S}$ and $\mathbb{I}_{\mathcal{S}}(x) = 0$ if $x \notin \mathcal{S}$. $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity function.

A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* if $\limsup_{i\to\infty} \phi(x_i) \leq \phi(x)$ whenever $\lim_{i\to\infty} x_i = x$. The function $\mathbb{I}_{\mathcal{S}}$ is upper semicontinuous for closed $\mathcal{S}$. The domain of a set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is taken to be the set $\mathrm{dom}M := \{x \in \mathbb{R}^n \mid M(x) \neq \varnothing\}$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* if, for each $(x_i, y_i) \to (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying $y_i \in M(x_i)$ for all $i \in \mathbb{Z}_{\geq 0}$, $y \in M(x)$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is locally bounded at point $x \in \mathbb{R}^n$ if for some neighborhood $V \in \mathcal{N}(x)$ the set $M(V) \subset \mathbb{R}^m$ is bounded; M is called *locally bounded* if this holds at every $x \in \mathbb{R}^n$ (Rockafellar & Wets, 1998, Definition 5.14). M is locally bounded if and only if for each bounded set $\mathcal{S} \in \mathbb{R}^n$, $M(\mathcal{S}) := \bigcup_{x\in\mathcal{S}} M(x)$ is bounded (Rockafellar & Wets, 1998, Proposition 5.15). A function $f : \mathcal{X} \to \mathcal{Y}$, with $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \subseteq \mathbb{R}^m$, is locally bounded if its set-valued extension $F(x) := \{f(x)\}$ for all $x \in \mathcal{X} = \mathrm{dom}f$, and $F(x) := \varnothing$ for all $x \notin \mathcal{X} = \mathrm{dom}f$, is locally bounded.

$\mathcal{B}(\mathbb{R}^n)$ denotes the Borel field, the subsets of $\mathbb{R}^n$ generated from open subsets of $\mathbb{R}^n$ through complements and finite and countable unions. A set $F \subset \mathbb{R}^n$ is *measurable* if $F \in \mathcal{B}(\mathbb{R}^n)$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *measurable* (Rockafellar & Wets, 1998, Definition 14.1) if for each open set $\mathcal{O} \subset \mathbb{R}^m$ the set $M^{-1}(\mathcal{O}) := \{v \in \mathbb{R}^n \mid M(v) \cap \mathcal{O} \neq \varnothing\}$ is measurable. When the values of M are closed, measurability is equivalent to $M^{-1}(\mathcal{C})$ being measurable for each closed set $\mathcal{C} \in \mathbb{R}^m$ (Rockafellar & Wets, 1998, Theorem 14.3).

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is *of class* $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and is unbounded. Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, a function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{PD}(\mathcal{A})$ if it is continuous and $\rho(x) = 0$ if and only if $x \in \mathcal{A}$.

3. Stochastic discrete-time systems

Consider a function $f : \mathcal{X} \times \mathcal{U} \times \mathcal{V} \to \mathcal{X}$, where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are closed sets, $\mathcal{V} \subseteq \mathbb{R}^p$ is measurable, and a stochastic controlled difference equation

$$x^+ = f(x, u, v) \tag{1}$$

with state variable $x \in \mathcal{X}$, control input $u \in \mathcal{U}$, and random input $v \in \mathcal{V}$, eventually specified as a random variable, that is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{V}$. From an infinite sequence of independent, identically distributed (i.i.d.) random variables $\mathbf{v}_i : \Omega \to \mathcal{V}$, for $i \in \mathbb{Z}_{\geq 0}$, a distribution function $\mu : \mathcal{B}(\mathcal{V}) \to [0, 1]$ defined as $\mu(F) := \mathbb{P}(\{\omega \in \Omega \mid \mathbf{v}_i(\omega) \in F\})$ is derived.

We consider the following regularity conditions throughout the paper.

Standing Assumption 1. The function f satisfies the following properties:

(1) f is locally bounded;
(2) for any $v \in \mathcal{V}$, the mapping $(x, u) \mapsto f(x, u, v)$ is continuous;
(3) for any $(x, u) \in \mathcal{X} \times \mathcal{U}$, the mapping $v \mapsto f(x, u, v)$ is measurable.

The first and the second conditions play a fundamental role in our robustness results. The third condition guarantees that the integrals in the paper are well defined for each fixed (x, u). As explained later on, it guarantees that f generates stochastic processes when u is a measurable function of x.

Given a stochastic difference equation of the kind

$$x^+ = g(x, v) \tag{2}$$

with $g : \mathcal{X} \times \mathcal{V} \to \mathcal{X}$ locally bounded, and $v \mapsto g(x, v)$ measurable for all $x \in \mathcal{X}$, let us define the notion of stochastic Lyapunov function.

Definition 1 (*SLF*). An upper semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *Stochastic Lyapunov Function* relative to the compact set $\mathcal{A} \subset \mathbb{R}^n$ for (2) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{PD}(\mathcal{A})$ such that for all $x \in \mathbb{R}^n$ we have $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$ and[2]

$$\int_{\mathcal{V}} V(g(x, v))\mu(dv) \leq V(x) - \rho(x). \tag{3}$$

In the main results of the paper, we will assume that there exists a locally bounded, possibly discontinuous, state-feedback control law, associated with a continuous stochastic Lyapunov function as follows.

[2] Since $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is upper semicontinuous and $v \mapsto g(x, v)$ is measurable, then $v \mapsto V(g(x, v))$ is measurable as well (Rockafellar & Wets, 1998, Theorem 14.13 (b)) and in turn the integral is well defined.